# A bivariate Hawkes process based model, for interest rates

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**Abstract**

Abstract: This paper proposes a continuous time model for interest rates, based on a bivariate self exciting point process. The two components of this process represent the global supply and demand for fixed income instruments. In this framework, closed form expressions are obtained for the first moments of the short term rate and for bonds, under an equivalent affine risk neutral measure. European derivatives are priced under a forward measure and a numerical algorithm is proposed to evaluate caplets and floorlets. The model is fitted to the time series of one year swap rates, from 2004 to 2014. From observation of yield curves over the same period, we filter the evolution of risk premiums of supply and demand processes. Finally, we analyze the sensitivity of implied volatilities of caplets to parameters defining the level of mutual-excitation.

Keywords. Hawkes process, self-exciting process, interest rate, micro-structure, yield curve.

## 1 Introduction

During the recent crisis of European sovereign debts, fixed income markets collapsed and caused liquidity shortfalls in countries of South Europa. The immediacy of information contributed to speed up the tightening of traded volumes of short and long term bonds. And the abrupt decline in demand for debts, due to the anxiety about excessive national debt, even if correlated with a reduction of supply, raised interest rates to historical summits, in Greece (33.7% for the 10 year bond on the 3/2/2012), Italy, Spain and Portugal. On another side,by the end of 2011, Germany was estimated to have made more than €9 billion out of the crisis as investors flocked to safer but near zero interest rate German federal government bonds. By July 2012 the Netherlands, Austria and Finland also benefited from zero or negative interest rates, as consequence of the high demand for their national debt. This crisis reminds us that interest rates basically depend on the law of supply and demand. There is also compelling evidence that yields of fixed income instruments are affected by liquidity concerns, as shown by Longstaff (2004), Landschoot (2008), Chen, Lesmond and Wei (2007), Covitz and Downing (2007) or Acharya and Pedersen (2005). Understanding liquidity effects in bonds markets is then of particular importance for central banks, to define appropriate monetary policy actions.

As liquidity shortages result from a disequilibrium between the global demand and supply for debts, the model developed in this work assumes that the short term rate is ruled by two Hawkes processes, representing the aggregate bid and ask orders, for fixed income instruments. This approach is fully relevant with the monetary theory as e.g. detailed in the chapter 5 of Mishkin (2007), and presents several interesting features. Firstly, it introduces path dependency and auto-correlation, that are absent from models based on Brownian motions (Cox et al., 1985 ; Hull and White, 1990 ; Duffie and Kan, 1996; Dai and Singleton, 2000 ; Brigo and Mercurio, 2007, for a survey), on Lévy processes (Eberlein and Kluge, 2006; Filipović and Tappe, 2008; Hainaut and Macgilchrist, 2010) or on switching processes (Hainaut 2013 ; Shen and Siu 2013).

Secondly, it adds mutual excitation and snowball effects, between the supply and demand in interest rate markets. A Hawkes process (see Hawkes (1971a) (1971b), Hawkes and Oakes (1974)), is indeed a parsimonious self exciting point process for which the intensity jumps in response and reverts to a target level in the absence of event. As the future of a self exciting process is influenced by the timing of past events, Errais et al. (2010) use this to generate contagion between defaults in a top down approach to credit risk. Embrechts et al. (2011) applied multivariate Hawkes processes in their analysis of stocks markets. Hawkes processes are also used by Aït-Sahalia et al. (2014a), (2014b) to model two key aspects of asset prices: clustering in time and cross sectional contamination between regions. On another hand, these processes are increasingly integrated in high frequency finance. Examples include the modeling of the duration between trades (Bauwens and Hautsch, 2009) or the arrival process of buy and sell orders, as in Bacry et al. (2013). Giot (2005), Chavez-Demoulin et al. (2005) or Chavez-Demoulin and McGill (2012) test these processes in a risk management context. Whereas Dassios and Jang (2012) propose a bivariate process for applications in insurance.

This research complements the existing literature about interest rate modeling in several directions. It is one of the first to use exclusively a bivariate Hawkes process for the modeling of the term structure of interest rates. Secondly, the model is compliant with the theory of monetary economics. Thirdly, this work provides all the tools for pricing bonds and for reconciling the dynamics of the short term rate under the real measure, with the term structure of bonds yields, evaluated under the risk neutral measure. We propose a family of changes of measure that preserves the dynamics of the process under real and risk neutral measures. Finally, after an analyze of the dynamics of bond quotes, the moment generating function of bond yields under a forward measure is detailed and a discrete Fourier transform algorithm is proposed to price derivatives.

The paper proceeds as follows. Section 2 introduces the model and derives its main features, like the moments of intensities for the arrival of bid and ask orders. In Section 3, we present equivalent exponential affine measures and study the conditions ensuring that the equivalent measure is risk neutral. After a presentation of the dynamics of the short term rate under this risk neutral measure, a formula for bond pricing is proposed. The section 4 is about the valuation of derivatives. In section 5, we fit the model to the time series of one year swap rate. From observation of yield curves over the same period, we filter the evolution of risk premiums of supply and demand processes. Finally, we test the sensitivity of yield curves and smiles of implied volatilities to changes of parameters.

## 2 Model.

The short term interest rate, $r_t$ is assumed to be the sum of a function of time $\varphi(t)$ and of a process $X_t$,

$$r_t \quad = \quad \varphi(t) + X_t, \tag{1}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$, with a right-continuous information filtration $\mathbb{F} = (\mathcal{F}_t)_{t>0}$, where $P$ denotes the real probability measure. In most of affine models, the process $X_t$ is Gaussian. As our purpose is to emphasize the link existing between the level of interest rates and bid-ask orders for bonds or any other interest rate products, $X_t$ is defined as the difference between the total supply and demand for such instruments. On another hand, to introduce path dependency and mutual excitation between the arrivals of bid and ask orders, the aggregate supply and demand are modeled by a bivariate Hawkes process. These orders and their numbers are respectively noted $O^1$ ,$O^2$ and $N_t^1$ ,$N_t^2$. The processes modeling the aggregate supply and demand are then defined as the total of all bid and orders till time $t$:

$$L_t^1 = \sum_{i=1}^{N_t^1} O_i^1, \tag{2}$$

$$L_t^2 = \sum_{i=1}^{N_t^2} O_i^2. \tag{3}$$

The orders sizes, $O_t^1$ and $O_t^2$, are distributed on $(0, +\infty)$ according to $\nu_1(z)$ and $\nu_2(z)$. The positivity of $O_t^1$ and $O_t^2$ ensures the identifiability of the model. In the numerical illustration, order sizes are exponential random variables but any other type of positive distribution can be used. In later developments, their first and second moments are noted $\mu_1 = \mathbb{E}(O^1)$, $\mu_2 = \mathbb{E}(O^2)$, $\eta_1 = \mathbb{E}\left((O^1)^2\right)$, $\eta_2 = \mathbb{E}\left((O^2)^2\right)$. From the economic theory (see e.g. chapter 5 of Mishkin (2007) ), we know that an increase of the aggregate offer, $L_t^1$, of bonds causes a decline of their prices and a rise of interest rates. In the opposite scenario, under the pressure of a high aggregate demand $L_t^2$, bonds prices grow up and interest rates drop. Then if $\alpha_1$ and $\alpha_2$ respectively denotes the permanent impact of sell and buy orders of bonds, the economic theory suggests the following dynamics for $X_t$:

$$X_t = \alpha_1 L_t^1 - \alpha_2 L_t^2 \tag{4}$$

and its differential dynamics is given by:

$$\begin{aligned} dX_t &= \alpha_1 dL_t^1 - \alpha_2 O^2 dL_t^2 \\ &= \alpha_1 O^1 dN_t^1 - \alpha_2 O^2 dN_t^2 \end{aligned} \tag{5}$$

As $O_t^1$ and $O_t^2$ are positive, positive and negative variations of interest rates are respectively attributed to arrivals of bid and ask orders. The chosen dynamics for $r_t$ allows negative interest rates but we don't consider it as a limitation. Indeed, since the 2012 crisis of the European sovereign debts, we have observed several periods during which short term rates (sovereign or interbank) were negative (e.g. in 2014, the EONIA was negative 61 times over 254 days of trading). On another hand, the probability of observing negative rates can be restricted by an appropriate choice for the dynamics of $N_t^1$ ,$N_t^2$, as discussed later in section 4. The arrivals of buy or sell orders are point processes with self exciting dynamics and their intensities are random processes governed by the next equations:

$$d\lambda_t^i = \kappa_i(c_i - \lambda_t^i)dt + \delta_{i,1} dL_t^1 + \delta_{i,2} dL_t^2 \quad i = 1, 2, \tag{6}$$

where $\delta_{i,j}$ for $i, j = 1, 2$ are constant. Coefficients $\delta_{1,2}$ and $\delta_{2,1}$ set the cross impact of demand on supply and vice versa. They measure the dependence between them and can capture some interesting stylized facts like the impact of bond issuance during a period of low interest rates. E.g. if $\delta_{12} > 0$, the frequency of bonds issuance increases when the demand, $L_t^2$, steps up and drives down interest rates according to equation (5).

As shown in Errais et al. (2010), if $J_t^i = (L_t^i, N_t^i)$, the process $(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2)$ is a Markov process in the state space $D = (\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N})^2$ and its infinitesimal generator for any function $g : D \to \mathbb{R}$ with partial derivatives $g_{\lambda_1}$ , $g_{\lambda_2}$, is such that:

$$\begin{aligned} \mathcal{A}g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2) &= \kappa_1(c_1 - \lambda_t^1) g_{\lambda^1} + \kappa_2(c_2 - \lambda_t^2) g_{\lambda^2} \\ &+\lambda_t^1 \int_{-\infty}^{+\infty} g(\lambda_t^1 + \delta_{1,1} z, J_t^1 + (z,1)^\top, \lambda_t^2 + \delta_{2,1} z, J_t^2) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2) d\nu_1(z) \\ &+\lambda_t^2 \int_{-\infty}^{+\infty} g(\lambda_t^1 + \delta_{1,2} z, J_t^1, \lambda_t^2 + \delta_{2,2} z, J_t^2 + (z,1)^\top) - g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2) d\nu_2(z). \end{aligned} \tag{7}$$

Under mild conditions, the expectation of $g(.)$ is equal to the integral of the expected infinitesimal generator:

$$
\begin{aligned}
\mathbb{E}\left(g(\lambda_T^1, J_T^1, \lambda_T^2, J_T^2)|\mathcal{F}_t\right) &= g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2) + \mathbb{E}\left(\int_t^T \mathcal{A}g(\lambda_s^1, J_s^1, \lambda_s^2, J_s^2)ds|\mathcal{F}_t\right) \\
&= g(\lambda_t^1, J_t^1, \lambda_t^2, J_t^2) + \int_t^T \mathbb{E}\left(\mathcal{A}g(\lambda_s^1, J_s^1, \lambda_s^2, J_s^2)|\mathcal{F}_t\right) ds.
\end{aligned} \tag{8}
$$

The derivative of this expectation with respect to time is equal to its expected infinitesimal generator:

$$
\frac{\partial}{\partial T}\mathbb{E}\left(g(\lambda_T^1, J_T^1, \lambda_T^2, J_T^2)|\mathcal{F}_t\right) = \mathbb{E}\left(\mathcal{A}g(\lambda_T^1, J_T^1, \lambda_T^2, J_T^2)|\mathcal{F}_t\right), \tag{9}
$$

The next proposition relies on this last feature to calculate the first moments of intensities.

**Proposition 2.1.** *Let $m_i(t)$ denote the expected intensity*[1]*, $\mathbb{E}\left(\lambda_t^i\right)$, for i=1,2. They are given by the following expressions*

$$
\begin{aligned}
\begin{pmatrix} m_1(t) \\ m_2(t) \end{pmatrix} &= V \begin{pmatrix} \frac{1}{\gamma_1}\left(e^{\gamma_1 t} - 1\right) & 0 \\ 0 & \frac{1}{\gamma_2}\left(e^{\gamma_2 t} - 1\right) \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} \\
&+ V \begin{pmatrix} e^{\gamma_1 t} & 0 \\ 0 & e^{\gamma_2 t} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_0^1 \\ \lambda_0^2 \end{pmatrix},
\end{aligned} \tag{10}
$$

*where $\gamma_{1,2}$ are constant,*

$$
\begin{aligned}
\gamma_{1,2} := &\ \frac{1}{2}\left((\delta_{1,1}\mu_1 - \kappa_1) + (\delta_{2,2}\mu_2 - \kappa_2)\right) \pm \\
&\ \frac{1}{2}\sqrt{\left((\delta_{1,1}\mu_1 - \kappa_1) - (\delta_{2,2}\mu_2 - \kappa_2)\right)^2 + 4\delta_{1,2}\delta_{2,1}\mu_1\mu_2},
\end{aligned} \tag{11}
$$

*$V$ ,$V^{-1}$ are matrix given by:*

$$
V = \begin{pmatrix} -\delta_{1,2}\mu_2 & -\delta_{1,2}\mu_2 \\ (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_1 & (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_2 \end{pmatrix} \tag{12}
$$

$$
V^{-1} = \frac{1}{\Upsilon}\begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_2 & \delta_{1,2}\mu_2 \\ \gamma_1 - (\delta_{1,1}\mu_1 - \kappa_1) & -\delta_{1,2}\mu_2 \end{pmatrix} \tag{13}
$$

*and $\Upsilon$ is defined by*

$$
\Upsilon := -\delta_{1,2}\mu_2\sqrt{\left((\delta_{1,1}\mu_1 - \kappa_1) - (\delta_{2,2}\mu_2 - \kappa_2)\right)^2 + 4\delta_{1,2}\delta_{2,1}\mu_1\mu_2}. \tag{14}
$$

*Proof.* Consider the functions $g^i = \lambda_t^i$ for $i = 1, 2$. According to equations (7) and (8), their expectations are such that

$$
\begin{aligned}
\mathbb{E}(\mathcal{A}g^1) &= \kappa_1(c_1 - \mathbb{E}\left(\lambda_t^1\right))dt + \mathbb{E}\left(\lambda_t^1\right)\int_{-\infty}^{+\infty}\delta_{1,1}z\,d\nu_1(z)dt + \mathbb{E}\left(\lambda_t^2\right)\int_{-\infty}^{+\infty}\delta_{1,2}z\,d\nu_2(z)dt \\
&= \kappa_1(c_1 - \mathbb{E}\left(\lambda_t^1\right))dt + \mathbb{E}\left(\lambda_t^1\right)\delta_{1,1}\mu_1 dt + \mathbb{E}\left(\lambda_t^2\right)\delta_{1,2}\mu_2 dt \\
\mathbb{E}(\mathcal{A}g^2) &= \kappa_2(c_2 - \mathbb{E}\left(\lambda_t^2\right))dt + \mathbb{E}\left(\lambda_t^1\right)\int_{-\infty}^{+\infty}\delta_{2,1}z\,d\nu_1(z)dt + \mathbb{E}\left(\lambda_t^2\right)\int_{-\infty}^{+\infty}\delta_{2,2}z\,d\nu_2(z)dt \\
&= \kappa_2(c_2 - \mathbb{E}\left(\lambda_t^2\right))dt + \mathbb{E}\left(\lambda_t^1\right)\delta_{2,1}\mu_1 dt + \mathbb{E}\left(\lambda_t^2\right)\delta_{2,2}\mu_2 dt
\end{aligned}
$$

[1]Remark that $\mathbb{E}\left(.|\mathcal{F}_0\right)$ is abusively denoted by $\mathbb{E}\left(.\right)$ in later developments.

If we refer to equation (9), moments $m_1(t)$ and $m_2(t)$ are solutions of a system of ordinary differential equations (ODEs) with respect to time:

$$\frac{\partial}{\partial t}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} + \begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2 - \kappa_2) \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}. \tag{15}$$

Finding a solution requires to determine eigenvalues $\gamma$ and eigenvectors $(v_1, v_2)$ of the matrix present in the right term of this system:

$$\begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2 - \kappa_2) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \gamma \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Eigenvalues cancel the determinant of the following matrix:

$$\det\begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) - \gamma & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2 - \kappa_2) - \gamma \end{pmatrix} = 0$$

and are solutions of the second order equation:

$$\gamma^2 - \gamma\left((\delta_{1,1}\mu_1 - \kappa_1) + (\delta_{2,2}\mu_2 - \kappa_2)\right) + (\delta_{1,1}\mu_1 - \kappa_1)(\delta_{2,2}\mu_2 - \kappa_2) - \delta_{1,2}\delta_{2,1}\mu_1\mu_2 = 0$$

Roots of this last equation are $\gamma_1$ and $\gamma_2$, as defined by the equation (11). One way to find an eigenvector is to note that it must be orthogonal to each rows of the matrix:

$$\begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) - \gamma & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2 - \kappa_2) - \gamma \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

then necessary,

$$\begin{pmatrix} v_1^i \\ v_2^i \end{pmatrix} = \begin{pmatrix} -\delta_{1,2}\mu_2 \\ (\delta_{1,1}\mu_1 - \kappa_1) - \gamma_i \end{pmatrix} \quad for\ i = 1, 2.$$

If $D = diag(\gamma_1, \gamma_2)$. The matrix in the right term of equation (15) admits the representation:

$$\begin{pmatrix} (\delta_{1,1}\mu_1 - \kappa_1) & \delta_{1,2}\mu_2 \\ \delta_{2,1}\mu_1 & (\delta_{2,2}\mu_2 - \kappa_2) \end{pmatrix} = VDV^{-1},$$

where $V$ is the matrix of eigenvectors, as defined in equation (12). Its determinant, $\Upsilon$, and its inverse are respectively provided by equations (14) and (13). If two new variables are defined as follows:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = V^{-1}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

The system (15) is decoupled into two independent ODEs:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = V^{-1}\begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{16}$$

And introducing the following notations

$$V^{-1}\begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix},$$

leads to the solutions for the system (16):

$$\begin{aligned} u_1(t) &= \frac{\epsilon_1}{\gamma_1}\left(e^{\gamma_1 t} - 1\right) + d_1 e^{\gamma_1 t} \\ u_2(t) &= \frac{\epsilon_2}{\gamma_2}\left(e^{\gamma_2 t} - 1\right) + d_2 e^{\gamma_2 t} \end{aligned}$$

where $d = (d_1, d_2)'$ is such that $d = V^{-1}\lambda_0$. Or in in matrix form,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma_1}\left(e^{\gamma_1 t}-1\right) & 0 \\ 0 & \frac{1}{\gamma_2}\left(e^{\gamma_2 t}-1\right) \end{pmatrix} V^{-1} \begin{pmatrix} \kappa_1 c_1 \\ \kappa_2 c_2 \end{pmatrix} + \begin{pmatrix} e^{\gamma_1 t} & 0 \\ 0 & e^{\gamma_2 t} \end{pmatrix} V^{-1} \begin{pmatrix} \lambda_0^1 \\ \lambda_0^2 \end{pmatrix}$$

Expressions (10) for $m_1, m_2$ are inferred from this last relation. □

The next corollary is an immediate consequence of this last proposition.

**Corollary 2.2.** *The expectation of $X_t$ is equal to:*

$$\mathbb{E}\left(X_t\right) = X_0 + \alpha_1\mu_1 \int^t m_1(s)ds - \alpha_2\mu_2 \int_0^t m_2(s)ds \tag{17}$$

*Proof.* Let us denote $f = \mathbb{E}\left(X_t\right)$, then the expectation of its infinitesimal generator is equal to

$$\begin{aligned} \mathbb{E}(\mathcal{A}f) &= \mathbb{E}\left(\lambda_t^1\right) \int_{-\infty}^{+\infty} \alpha_1 z \, d\nu_1(z)dt - \mathbb{E}\left(\lambda_t^2\right) \int_{-\infty}^{+\infty} \alpha_2 z \, d\nu_2(z)dt \\ &= \mathbb{E}\left(\lambda_t^1\right)\alpha_1\mu_1 dt - \mathbb{E}\left(\lambda_t^2\right)\alpha_2\mu_2 dt \end{aligned}$$

and according to equation (8), we conclude. □

The system of ODEs that rules variances and correlation of intensities is provided in the next proposition.

**Proposition 2.3.** *Let us denote the variance of $\lambda_i$ by $V_i(t) = \mathbb{E}\left(\left(\lambda_t^i\right)^2\right) - \left(m_i(t)\right)^2$ for $i = 1, 2$ and their covariance by $V_3(t) = \mathbb{E}\left(\lambda_t^1\lambda_t^2\right) - m_1(t)m_2(t)$. They are solutions of the following system of ODEs:*

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = & \begin{pmatrix} \delta_{1,1}^2\eta_1 & \delta_{1,2}^2\eta_2 \\ \delta_{2,1}^2\eta_1 & \delta_{2,2}^2\eta_2 \\ \delta_{1,1}\delta_{2,1}\eta_1 & \delta_{2,2}\delta_{1,2}\eta_2 \end{pmatrix} \begin{pmatrix} m_1(t) \\ m_2(t) \end{pmatrix} + \\ & \begin{pmatrix} 2\left(\delta_{1,1}\mu_1 - \kappa_1\right) & 0 & 2\delta_{1,2}\mu_2 \\ 0 & 2\left(\delta_{2,2}\mu_2 - \kappa_2\right) & 2\delta_{2,1}\mu_1 \\ \delta_{2,1}\mu_1 & \delta_{1,2}\mu_2 & \left(\delta_{1,1}\mu_1 + \delta_{2,2}\mu_2\right) - \kappa_1 - \kappa_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}, \end{aligned} \tag{18}$$

*with initial conditions*

$$V_i(0) = 0 \qquad \text{for } i = 1, 2, 3.$$

*Proof.* Let us introduce the following notations: $g_i = \left(\lambda_t^i\right)^2$ for $i = 1, 2$ and $g_3 = \lambda_t^1\lambda_t^2$, according to equations (14) and (13), the next relation holds

$$\begin{aligned} \mathcal{A}g_1 &= 2\lambda_t^1\kappa_1 c_1 - 2\left(\lambda_t^1\right)^2\kappa_1 dt + \lambda_t^1 \int_{-\infty}^{+\infty} 2\lambda_t^1\delta_{1,1}z + \left(\delta_{1,1}z\right)^2 d\nu_1(z)dt \\ &\quad + \lambda_t^2 \int_{-\infty}^{+\infty} 2\lambda_t^1\delta_{1,2}z + \left(\delta_{1,2}z\right)^2 d\nu_2(z)dt \\ \mathcal{A}g_2 &= 2\lambda_t^2\kappa_2 c_2 - 2\left(\lambda_t^2\right)^2\kappa_2 dt + \lambda_t^2 \int_{-\infty}^{+\infty} 2\lambda_t^2\delta_{2,2}z + \left(\delta_{2,2}z\right)^2 d\nu_2(z)dt \\ &\quad + \lambda_t^1 \int_{-\infty}^{+\infty} 2\lambda_t^2\delta_{2,1}z + \left(\delta_{2,1}z\right)^2 d\nu_1(z)dt \\ \mathcal{A}g_3 &= \kappa_1(c_1 - \lambda_t^1)\lambda_t^2 + \kappa_2(c_2 - \lambda_t^2)\lambda_t^1 \\ &\quad + \lambda_t^1 \int_{-\infty}^{+\infty} \left(\lambda_t^1 + \delta_{1,1}z\right)\left(\lambda_t^2 + \delta_{2,1}z\right) - \lambda_t^1\lambda_t^2 \, d\nu_1(z)dt \\ &\quad + \lambda_t^2 \int_{-\infty}^{+\infty} \left(\lambda_t^1 + \delta_{1,2}z\right)\left(\lambda_t^2 + \delta_{2,2}z\right) - \lambda_t^1\lambda_t^2 \, d\nu_2(z)dt \end{aligned}$$

If we note $v_i = \mathbb{E}\left(\left(\lambda_t^i\right)^2\right)$ for $i = 1, 2$ and $v_3 = \mathbb{E}\left(\lambda_t^1 \lambda_t^2\right)$, a system of ODEs is deduced from equation (9):

$$\begin{aligned}
\frac{\partial}{\partial t} v_1 &= 2m_1(t)\kappa_1 c_1 - 2v_1(t)\kappa_1 + 2v_2(t)\delta_{1,1}\mu_1 + \\
&\quad m_1(t)\delta_{1,1}^2\eta_1 + 2v_3(t)\delta_{1,2}\mu_2 + m_2(t)\delta_{1,2}^2\eta_2 \\
\\
\frac{\partial}{\partial t} v_2 &= 2m_2(t)\kappa_2 c_2 - 2v_2(t)\kappa_2 + 2v_2(t)\delta_{2,2}\mu_2 + \\
&\quad m_2(t)\delta_{2,2}^2\eta_2 + 2v_3(t)\delta_{2,1}\mu_1 + m_1(t)\delta_{2,1}^2\eta_1 \\
\\
\frac{\partial}{\partial t} v_3 &= m_2(t)\kappa_1 c_1 - \kappa_1 v_3(t) + m_1(t)\kappa_2 c_2 - \kappa_2 v_3(t) \\
&\quad + v_1(t)\delta_{2,1}\mu_1 + v_3(t)\delta_{1,1}\mu_1 + m_1(t)\delta_{1,1}\delta_{2,1}\eta_1 \\
&\quad + v_2(t)\delta_{1,2}\mu_2 + v_3(t)\delta_{2,2}\mu_2 + m_2(t)\delta_{2,2}\delta_{1,2}\eta_2
\end{aligned} \tag{19}$$

As centered second moments $V_i(t)$, are linked to non centered ones, $v_i$ by the next differential equations

$$\begin{cases} \frac{\partial}{\partial t} V_i = \frac{\partial}{\partial t} v_i - 2m_i \frac{\partial}{\partial t} m_i & i = 1, 2 \\ \frac{\partial}{\partial t} V_3 = \frac{\partial}{\partial t} v_3 - m_1 \frac{\partial}{\partial t} m_2 - m_2 \frac{\partial}{\partial t} m_1 & \end{cases}$$

It is sufficient to combine equations (15) and (19) to conclude. □

The next proposition presents the moment generating function of $X_t$ and of its integral. This result is used later to infer the price of a bond and its dynamics under an equivalent measure.

**Proposition 2.4.** *Let $\psi_1(.)$ and $\psi_2(.)$ denote the moment generating functions of $O^1$ and $O^2$:*

$$\psi_i(w) := \mathbb{E}\left(e^{wO^i}\right) \quad i = 1, 2. \tag{20}$$

*The moment generating function of $w_0 X_T - w_1 \int_t^T X_s ds + \begin{pmatrix} w_2 \\ w_3 \end{pmatrix}^\top \begin{pmatrix} \lambda_T^1 \\ \lambda_T^2 \end{pmatrix}$ is an affine function of $X_t$ and of intensities:*

$$\mathbb{E}\left( e^{w_0 X_T - w_1 \int_t^T X_s ds + \begin{pmatrix} w_2 \\ w_3 \end{pmatrix}^\top \begin{pmatrix} \lambda_T^1 \\ \lambda_T^2 \end{pmatrix}} \mid \mathcal{F}_t \right) = \tag{21}$$

$$\exp\left( \left(w_0 - w_1(T-t)\right) X_t + A(t,T) + \begin{pmatrix} B_1(t,T) \\ B_2(t,T) \end{pmatrix}^\top \begin{pmatrix} \lambda_t^1 \\ \lambda_t^2 \end{pmatrix} \right)$$

*where $A(t,T)$ , $B_1(t,T)$ and $B_2(t,T)$ are solutions of a system of ODEs :*

$$\begin{cases} \frac{\partial}{\partial t} B_1(t,T) = \kappa_1 B_1(t,T) - \left[\psi_1\left(B_1(t,T)\delta_{1,1} + w_0\alpha_1 - w_1\alpha_1(T-t) + B_2(t,T)\delta_{2,1}\right) - 1\right] \\ \frac{\partial}{\partial t} B_2(t,T) = \kappa_2 B_2(t,T) - \left[\psi_2\left(B_1(t,T)\delta_{1,2} - w_0\alpha_2 + w_1\alpha_2(T-t) + B_2(t,T)\delta_{2,2}\right) - 1\right] \\ \frac{\partial}{\partial t} A(t,T) = -\kappa_1 c_1 B_1(t,T) - \kappa_2 c_2 B_2(t,T) \end{cases} \tag{22}$$

*with the terminal conditions $A(T,T) = 0$ , $B_1(T,T) = w_2$, $B_2(T,T) = w_3$.*

*Proof.* Let us define $Y_t := \mathbb{E}\left( e^{w_0 X_T - w_1 \int_t^T X_s ds + \left(\begin{array}{c} w_2 \\ w_3 \end{array}\right)^\top \left(\begin{array}{c} \lambda_T^1 \\ \lambda_T^2 \end{array}\right)} \,|\, \mathcal{F}_t \right)$. As $\mathcal{F}_t \subset \mathcal{F}_u$ for all $u \geq t$, the rule of conditional expectation states that

$$\begin{aligned} Y_t &= \mathbb{E}\left( e^{-w_1 \int_t^u X_s ds} \mathbb{E}\left( e^{w_0 X_T - w_1 \int_u^T X_s ds + \left(\begin{array}{c} w_2 \\ w_3 \end{array}\right)^\top \left(\begin{array}{c} \lambda_T^1 \\ \lambda_T^2 \end{array}\right)} \,|\, \mathcal{F}_u \right) \,|\, \mathcal{F}_t \right) \\ &= \mathbb{E}\left( e^{-w_1 \int_t^u X_s ds} Y_u \,|\, \mathcal{F}_t \right) \end{aligned}$$

Then, by assuming enough regularity to allow one to take the limit within the expectation, the following limit converges to zero:

$$\lim_{u \to t} \frac{\mathbb{E}\left( e^{-w_1 \int_t^u X_s ds} Y_u \,|\, \mathcal{F}_t \right) - Y_t}{u - t} = 0.$$

If we develop the exponential by its Taylor approximation of first order, we can rewrite this limit as:

$$\lim_{u \to t} \frac{\mathbb{E}\left( Y_u \,|\, \mathcal{F}_t \right) - Y_t}{u - t} = w_1 X_t Y_t \,. \tag{23}$$

The right hand term in this last equation is precisely the infinitesimal generator of $Y_t := f(t, \lambda_t^1, J_t^1, \lambda_t^2, J_t^2)$. If $f_t$, $f_{\lambda^1}$ and $f_{\lambda^2}$ denote respectively the partial derivatives of $f$ with respect to time and intensities, the equation (23) is rewritten as follows:

$$\begin{aligned} & w_1 \left( \alpha_1 L_t^1 - \alpha_2 L_t^2 \right) f = f_t + \kappa_1 (c_1 - \lambda_t^1) f_{\lambda^1} + \kappa_2 (c_2 - \lambda_t^2) f_{\lambda^2} \\ & + \lambda_t^1 \int_{-\infty}^{+\infty} f(t, \lambda_t^1 + \delta_{1,1} z, J_t^1 + (z,1)^\top, \lambda_t^2 + \delta_{2,1} z, J_t^2) - f(t, \lambda_t^1, J_t^1, \lambda_t^2, J_t^2) d\nu_1(z) \\ & + \lambda_t^2 \int_{-\infty}^{+\infty} f(t, \lambda_t^1 + \delta_{1,2} z, J_t^1, \lambda_t^2 + \delta_{2,2} z, J_t^2 + (z,1)^\top) - f(t, \lambda_t^1, J_t^1, \lambda_t^2, J_t^2) d\nu_2(z). \end{aligned} \tag{24}$$

$f$ also satisfies the next limit condition

$$f(T, \lambda_T^1, J_T^1, \lambda_T^2, J_T^2) = \exp\left( w_0 \left( \alpha_1 L_T^1 - \alpha_2 L_T^2 \right) + \left(\begin{array}{c} w_2 \\ w_3 \end{array}\right)^\top \left(\begin{array}{c} \lambda_T^1 \\ \lambda_T^2 \end{array}\right) \right) \tag{25}$$

Let us assume that $f$ has an exponential form

$$f \;=\; exp\left( A(t,T) + B(t,T)^\top \left(\begin{array}{c} \lambda_t^1 \\ \lambda_t^2 \end{array}\right) + C(t,T)^\top \left(\begin{array}{c} L_t^1 \\ L_t^2 \end{array}\right) \right)$$

where $B(t,T)^\top = (B_1(t,T)\,,\, B_2(t,T))^\top$ and $C(t,T)^\top = (C_1(t,T)\,,\, C_2(t,T))^\top$. Under this assumption, equation (24) becomes

$$\begin{aligned} & w_1 \left( \alpha_1 L_t^1 - \alpha_2 L_t^2 \right) = \left( \frac{\partial}{\partial t} A + \lambda_t^1 \frac{\partial}{\partial t} B_1 + \lambda_t^2 \frac{\partial}{\partial t} B_2 + L_t^1 \frac{\partial}{\partial t} C_1 + L_t^2 \frac{\partial}{\partial t} C_2 \right) \\ & \quad + \lambda_t^1 \left[ \left( (\psi_1 (B_1 \delta_{1,1} + C_1 + B_2 \delta_{2,1}) \right) - 1 \right] + \kappa_1 (c_1 - \lambda_t^1) B_1 \\ & \quad + \lambda_t^2 \left[ \left( \psi_2 \left( B_1 \delta_{1,2} + C_2 + B_2 \delta_{2,2} \right) - 1 \right) \right] + \kappa_2 (c_2 - \lambda_t^2) B_2. \end{aligned} \tag{26}$$

As $\lambda_t^i$ and $L_t^i$ are random for $i$=1,2, this last relation holds only if their multiplicative coefficients are null. This is achieved only if

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}B_1 - \kappa_1 B_1 + \left[\psi_1\left(B_1\delta_{1,1} + C_1 + B_2\delta_{2,1}\right) - 1\right], \\
0 &= \frac{\partial}{\partial t}B_2 - \kappa_2 B_2 + \left[\psi_2\left(B_1\delta_{1,2} + C_2 + B_2\delta_{2,2}\right) - 1\right], \\
0 &= \frac{\partial}{\partial t}A + \kappa_1 c_1 B_1 + \kappa_2 c_2 B_2, \\
w_1\alpha_1 &= \frac{\partial}{\partial t}C_1 \quad - w_1\alpha_2 = \frac{\partial}{\partial t}C_2,
\end{aligned} \tag{27}$$

From the boundary condition (25), we infer that $C_1(t,T) = w_0\alpha_1 - w_1\alpha_1(T-t)$ and $C_2(t,T) = -w_0\alpha_2 + w_1\alpha_2(T-t)$.

$\square$

Notice that it is possible to compute the probability density function of $X_t$, by inverting the moment generating function (21), with the Discrete Fourier Transform of proposition 3.8, in section 4.

# 3 Equivalent exponential affine measures and bond pricing.

As the characteristic function of $X_t$ is an affine function of $\left(\lambda_t^1, \lambda_t^2, L_t^1, L_t^2\right)$, we study exponential affine changes of measure and show that the dynamics of interest rates is preserved under the new measure. These equivalent measures are induced by an exponential martingale of the form:

$$M_t(\theta_1,\theta_2) := \exp\left(\left(a_1(\theta_1,\theta_2),\, a_2(\theta_1,\theta_2)\right)\begin{pmatrix}\lambda_t^1\\ \lambda_t^2\end{pmatrix} + (\theta_1,\theta_2)\begin{pmatrix}L_t^1\\ L_t^2\end{pmatrix} - \varphi(\theta_1,\theta_2)t\right). \tag{28}$$

where $\theta_1$, $\theta_2 \in \mathbb{R}$ and are assimilated later to risk premiums. Zhang et al. (2009) use a similar change of measure to simulate rare events, of a one dimension Hawkes process but with constant jumps. In our framework, jumps are random and the affine change of measure modifies both frequencies and distributions of jumps. Before detailing this point, the next proposition introduces the necessary conditions that $(\theta_1,\theta_2)$ fulfill to guarantee that $M_t(\theta_1,\theta_2)$ is a local martingale.

**Proposition 3.1.** *If for any given couple of parameters $(\theta_1,\theta_2)$, there exist suitable solutions $a_1(\theta_1,\theta_2)$ and $a_2(\theta_1,\theta_2)$ for the system of equations*

$$\begin{cases}
a_1(\theta_1,\theta_2)\kappa_1 - \left(\psi_1(a_1(\theta_1,\theta_2)\delta_{1,1} + a_2(\theta_1,\theta_2)\delta_{2,1} + \theta_1) - 1\right) &= 0\\
a_2(\theta_1,\theta_2)\kappa_2 - \left(\psi_2(a_2(\theta_1,\theta_2)\delta_{2,2} + a_1(\theta_1,\theta_2)\delta_{1,2} + \theta_2) - 1\right) &= 0
\end{cases} \tag{29}$$

*where $\psi_i(w) = \mathbb{E}(e^{wO^i})$ for $i = 1,2$, and if $\varphi(\theta_1,\theta_2)$ is a linear combination of these solutions*

$$\varphi(\theta_1,\theta_2) = a_1(\theta_1,\theta_2)\kappa_1 c_1 + a_2(\theta_1,\theta_2)\kappa_2 c_2 \tag{30}$$

*then $M_t(\theta_1,\theta_2)$ is a local martingale.*

*Proof.* Let us denote by $Y_t$ the exponent of $M_t$:

$$Y_t = \left(a_1(\theta_1,\theta_2),\, a_2(\theta_1,\theta_2)\right)\begin{pmatrix}\lambda_t^1\\ \lambda_t^2\end{pmatrix} + (\theta_1,\theta_2)\begin{pmatrix}L_t^1\\ L_t^2\end{pmatrix} - \varphi(\theta_1,\theta_2)t \tag{31}$$

According to equation (6), its infinitesimal dynamics is given by

$$\begin{aligned}
dY_t = {} & a_1\kappa_1(c_1 - \lambda_t^1)dt + a_2\kappa_2(c_2 - \lambda_t^2)dt + \left(a_1\delta_{1,1} + a_2\delta_{2,1} + \theta_1\right)dL_t^1 \\
& + \left(a_2\delta_{2,2} + a_1\delta_{1,2} + \theta_2\right)dL_t^2 - \varphi(\theta_1,\theta_2)dt
\end{aligned}$$

In the remainder of this proof, the random measure of $O^i$ is noted $\chi^i(.)$ and is such that $O^i = \int_{-\infty}^{\infty} \chi^i(dz)$ for $i = 1, 2$. Applying the Ito's lemma for semi-martingales to $M_t$ leads to the next relation:

$$\begin{aligned} dM_t &= M_t dY_t + \frac{1}{2} M_t d\left[Y_t, Y_t\right]_t^c \\ &+ M_t \int_{-\infty}^{\infty} \left(e^{(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)z} - 1 - (a_1\delta_{1,1} + a_2\delta_{2,1} + \theta_1)\, z\right) \chi^1(dz) dN_t^1 \\ &+ M_t \int_{-\infty}^{\infty} \left(e^{(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)z} - 1 - (a_2\delta_{2,2} + a_1\delta_{1,2} + \theta_2)\, z\right) \chi^2(dz) dN_t^2 \end{aligned}$$

or equal to

$$\begin{aligned} dM_t \;=\; & M_t \left(a_1\kappa_1 c_1 + a_2\kappa_2 c_2 - \varphi\right) dt \\ & - M_t \lambda_t^1 \left(a_1\kappa_1 - \int_{-\infty}^{\infty} \left(e^{(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)z} - 1\right) \nu_1(dz)\right) dt \\ & - M_t \lambda_t^2 \left(a_2\kappa_2 - \int_{-\infty}^{\infty} \left(e^{(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)z} - 1\right) \nu_2(dz)\right) dt \\ & + M_t \int_{-\infty}^{\infty} \left(e^{(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)z} - 1\right) \left[\chi^1(dz) dN_t^1 - \lambda_t^1 \nu_1(dz) dt\right] \\ & + M_t \int_{-\infty}^{\infty} \left(e^{(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)z} - 1\right) \left[\chi^2(dz) dN_t^2 - \lambda_t^2 \nu_2(dz) dt\right] . \end{aligned}$$

Since the integrals with respect to $\chi^i(dz)dN_t^i - \lambda_t^i \nu_i(dz)dt$ are local martingales, $M_t$ is also a local martingale if and only if the following relations hold:

$$\begin{cases} a_1(\theta_1, \theta_2)\kappa_1 c_1 + a_2(\theta_1, \theta_2)\kappa_2 c_2 - \varphi(\theta_1, \theta_2) & = 0 \\ a_1(\theta_1, \theta_2)\kappa_1 - \int_{-\infty}^{\infty} \left(e^{(a_1(\theta_1,\theta_2)\delta_{1,1}+a_2(\theta_1,\theta_2)\delta_{2,1}+\theta_1)z} - 1\right) \nu_1(dz) & = 0 \\ a_2(\theta_1, \theta_2)\kappa_2 - \int_{-\infty}^{\infty} \left(e^{(a_2(\theta_1,\theta_2)\delta_{2,2}+a_1(\theta_1,\theta_2)\delta_{1,2}+\theta_2)z} - 1\right) \nu_2(dz) & = 0 \end{cases}$$

and these conditions are equivalent to equations (29) and (30) □

Assuming the existence of suitable solutions for the system (29), an equivalent measure $Q^{\theta_1,\theta_2}$ is defined by:

$$\left.\frac{dQ^{\theta_1,\theta_2}}{dP}\right|_{\mathcal{F}_t} = \frac{M_t(\theta_1, \theta_2)}{M_0(\theta_1, \theta_2)} \tag{32}$$

and may be used as risk neutral measure by investors. In this case, the dynamics of intensities and aggregate supply or demand is modified but is still a bivariate Hawkes process:

**Proposition 3.2.** *Let $N_t^{1,Q}$ and $N_t^{2,Q}$ be counting processes with respective intensities*

$$\begin{cases} \lambda_t^{1,Q} = \mathbb{E}\left(e^{(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)O^1}\right) \lambda_t^1 \\ \lambda_t^{2,Q} = \mathbb{E}\left(e^{(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)O^2}\right) \lambda_t^2 \end{cases} , \tag{33}$$

*under the equivalent measure $Q^{\theta_1,\theta_2}$. On another hand, if $O^{1,Q}$, $O^{2,Q}$ denotes random variables defined by the following characteristic functions*

$$\begin{cases} \psi_1^Q(z) := \mathbb{E}\left(e^{zO^{1,Q}}\right) & = \frac{\psi_1(z+(\delta_{1,1}a_1+\delta_{2,1}a_2+\theta_1))}{\psi_1(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)} \\ \psi_2^Q(z) := \mathbb{E}\left(e^{zO^{2,Q}}\right) & = \frac{\psi_2(z+(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2))}{\psi_2(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)} \end{cases} \tag{34}$$

*and if $L_t^{1,Q}$ , $L_t^{2,Q}$ are defined by the next jump processes*

$$L_t^{i,Q} \;=\; \sum_{k=1}^{N_t^{i,Q}} O_k^{i,Q} \quad i=1,2, \tag{35}$$

*intensities $\lambda_t^j$ are driven by the following SDE under $Q^{\theta_1,\theta_2}$*

$$d\lambda_t^i \;=\; \kappa_i(c_i-\lambda_t^i)dt+\delta_{i,1}dL_t^{1,Q}+\delta_{i,2}dL_t^{2,Q} \quad i=1,2. \tag{36}$$

*Proof.* If $Y_t$ is the exponent of $M_t$, as defined by equation (31), the characteristic function of $X_T$ under the risk neutral is then equal to

$$\begin{aligned}\mathbb{E}^Q\left(e^{wX_T}|\mathcal{F}_t\right) &= \mathbb{E}\left(e^{Y_T-Y_t+wX_T}|\mathcal{F}_t\right)\\ &= e^{-Y_t}\mathbb{E}\left(e^{Y_T+wX_T}|\mathcal{F}_t\right)\end{aligned}$$

If $f(t,\lambda_t^1,J_r^1,\lambda_r^2,J_r^2)$ denotes $\mathbb{E}\left(e^{Y_T+wX_T}|\mathcal{F}_t\right)$, according to the Itô's lemma, it solves the next equation

$$\begin{aligned}0 &= f_t+\kappa_1(c_1-\lambda_t^1)f_{\lambda^1}+\kappa_2(c_2-\lambda_t^2)f_{\lambda^2}\\ &+\lambda_t^1\int_{-\infty}^{+\infty} f(t,\lambda_t^1+\delta_{1,1}z,J_t^1+(z,1)^\top,\lambda_t^2+\delta_{2,1}z,J_t^2)-f(t,\lambda_t^1,J_t^1,\lambda_t^2,J_t^2)d\nu_1(z)\\ &+\lambda_t^2\int_{-\infty}^{+\infty} f(t,\lambda_t^1+\delta_{1,2}z,J_t^1,\lambda_t^2+\delta_{2,2}z,J_t^2+(z,1)^\top)-f(t,\lambda_t^1,J_t^1,\lambda_t^2,J_t^2)d\nu_2(z).\end{aligned} \tag{37}$$

where $f_t$, $f_{\lambda^1}$, $f_{\lambda^2}$ are the partial derivatives of $f(.)$ with respect to time and intensities. Furthermore given that

$$\begin{aligned}Y_T+wX_T &= a_1\left(\lambda_T^1-\kappa_1c_1T\right)+a_2\left(\lambda_T^2-\kappa_2c_2T\right)\\ &\quad+(\theta_1+\alpha_1w)\,L_T^1+(\theta_2+\alpha_2w)\,L_T^2\,.\end{aligned} \tag{38}$$

$f(.)$ satisfies the following terminal condition at time $t=T$:

$$\begin{aligned}f(T,\lambda_T^1,J_T^1,\lambda_T^2,J_T^2) &= \exp\left((\theta_1+\alpha_1w)\,L_T^1+(\theta_2+\alpha_2w)\,L_T^2\right.\\ &\quad\left.+a_1\left(\lambda_T^1-\kappa_1c_1T\right)+a_2\left(\lambda_T^2-\kappa_2c_2T\right)\right).\end{aligned}$$

In the remainder of this section, it is assumed that $f(.)$ is an exponential affine function:

$$f \;=\; exp\left(A(t,T)+B(t,T)^\top\begin{pmatrix}\psi_1(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)\lambda_t^1\\ \psi_2(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)\lambda_t^2\end{pmatrix}+C(t,T)^\top\begin{pmatrix}L_t^1\\ L_t^2\end{pmatrix}\right)$$

where $B(t,T,w)^\top=(B_1(t,T,w)\,,\,B_2(t,T,w))^\top$ and $C(t,T,w)^\top=(C_1(t,T,w)\,,\,C_2(t,T,w))^\top$. Under this assumption, the partial derivatives of $f$ are given by :

$$f_t \;=\; \left(\frac{\partial}{\partial t}A+\psi_1\lambda_t^1\frac{\partial}{\partial t}B_1+\psi_2\lambda_t^2\frac{\partial}{\partial t}B_2+L_t^1\frac{\partial}{\partial t}C_1+L_t^2\frac{\partial}{\partial t}C_2\right)f$$

$$f_{\lambda^1}=\psi_1B_1f \qquad f_{\lambda^2}=\psi_2B_2f$$

where $\psi_1$ and $\psi_2$ abusively denote $\psi_1(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)$ and $\psi_2(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)$. Integrands in equation (37) are equal to:

$$\begin{aligned}&f(t,\lambda_t^1+\delta_{1,1}z,J_t^1+(z,1)^\top,\lambda_t^2+\delta_{2,1}z,J_t^2)-f(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2)\\ &\quad=f\left[\exp\left((B_1\psi_1\delta_{1,1}+C_1+B_2\psi_2\delta_{2,1})\,z\right)-1\right],\\ &f(\lambda_t^1+\delta_{1,2}z,J_t^1,\lambda_t^2+\delta_{2,2}z,J_t^2+(z,1)^\top)-f(\lambda_t^1,J_t^1,\lambda_t^2,J_t^2)\\ &\quad=f\left[\exp\left((B_1\psi_1\delta_{1,2}+C_2+B_2\psi_2\delta_{2,2})\,z\right)-1\right].\end{aligned}$$

Injecting these expressions in equation (37) yields a system of ODEs for $A$, $B_1$ and $B_2$:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}B_1 - \kappa_1 B_1 + \frac{1}{\psi_1}\left[\psi_1\left(B_1\psi_1\delta_{1,1} + C_1 + B_2\psi_2\delta_{2,1}\right) - 1\right], \\
0 &= \frac{\partial}{\partial t}B_2 - \kappa_2 B_2 + \frac{1}{\psi_2}\left[\psi_2\left(B_1\psi_1\delta_{1,2} + C_2 + B_2\psi_2\delta_{2,2}\right) - 1\right], \\
0 &= \frac{\partial}{\partial t}A + \psi_1\kappa_1 c_1 B_1 + \psi_2\kappa_2 c_2 B_2, \\
0 &= \frac{\partial}{\partial t}C_1, \quad 0 = \frac{\partial}{\partial t}C_2.
\end{aligned} \tag{39}$$

with the terminal conditions:

$$\begin{cases} A(T,T) & = -a_1\kappa_1 c_1 T - a_2\kappa_2 c_2 T \\ B_1(T,T) & = \frac{a_1}{\psi_1} \quad B_2(T,T) = \frac{a_2}{\psi_2} \\ C_1(T,T) & = (\theta_1 + \alpha_1 w) \\ C_2(T,T) & = (\theta_2 + \alpha_2 w) \end{cases}.$$

As $C_1(t,T) = \theta_1 + \alpha_1 w$ and $C_2(t,T) = \theta_2 + \alpha_2 w$, the moment generating function of $X_t$ is equal to:

$$\begin{aligned}
\mathbb{E}^Q\left(e^{wX_T}|\mathcal{F}_t\right) &= e^{-Y_t}\mathbb{E}\left(e^{Y_T + wX_T}|\mathcal{F}_t\right) \\
&= \exp\left(A + a_1\kappa_1 c_1 t + a_2\kappa_2 c_2 t + (B_1\psi_1 - a_1)\lambda_t^1 + (B_2\psi_2 - a_2)\lambda_t^2 + wX_t\right).
\end{aligned}$$

In the remainder of the proof, this expectation is restated in a form similar to the moment generating function of $X_T$ under $P$. To achieve this, the following change of variables is done:

$$\begin{aligned}
A' &:= A + a_1\kappa_1 c_1 t + a_2\kappa_2 c_2 t, \\
B_1^{'} &:= B_1 - \frac{a_1}{\psi_1}, \\
B_2^{'} &:= B_2 - \frac{a_2}{\psi_2}.
\end{aligned}$$

with the terminal conditions $A^{'}(T,T) = 0$ , $B_1^{'}(T,T) = 0$, $B_2^{'}(T,T) = 0$. As from equation (29), the following relation holds

$$\begin{cases} \kappa_1\frac{a_1}{\psi_1} = \left(1 - \frac{1}{\psi_1}\right) \\ \kappa_2\frac{a_2}{\psi_2} = \left(1 - \frac{1}{\psi_2}\right) \end{cases} \tag{40}$$

The system of ODE's (39) becomes:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}B_1^{'} - \kappa_1 B_1^{'} + \left[\frac{1}{\psi_1}\psi_1\left(B_1^{'}\psi_1\delta_{1,1} + (\theta_1 + \delta_{1,1}a_1 + \delta_{2,1}a_2) + \alpha_1 w + B_2^{'}\psi_2\delta_{2,1}\right) - 1\right], \\
0 &= \frac{\partial}{\partial t}B_2^{'} - \kappa_2 B_2^{'} + \left[\frac{1}{\psi_2}\psi_2\left(B_1^{'}\psi_1\delta_{1,2} + (\theta_2 + \delta_{1,2}a_1 + \delta_{2,2}a_2) + \alpha_2 w + B_2^{'}\psi_2\delta_{2,2}\right) - 1\right], \\
0 &= \frac{\partial}{\partial t}A' + \psi_1\kappa_1 c_1 B_1^{'} + \psi_2\kappa_2 c_2 B_2^{'},
\end{aligned}$$

If we consider jumps $O^{1,Q}$, $O^{2,Q}$ that have moment generating functions defined by equations (34),the moment generating function of $X_T$ under $Q$ is given by

$$\mathbb{E}^Q\left(e^{wX_T}|\mathcal{F}_t\right) = \exp\left(wX_t + A^{'}(t,T) + \begin{pmatrix} B_1^{'}(t,T) \\ B_2^{'}(t,T) \end{pmatrix}^\top \begin{pmatrix} \lambda_t^{1,Q} \\ \lambda_t^{2,Q} \end{pmatrix}\right) \tag{41}$$

where $A^{'}$, $B_1^{'}$ and $B_2^{'}$ solve a system, identical to the one of proposition 2.4.

□

In numerical applications, sizes of orders under $P$ are exponential random variables and their probability density functions is defined by two parameters $\rho_1,\, \rho_2 \in \mathbb{R}^+$ as follows:

$$\nu_1(z) = \rho_1 e^{-\rho_1 z} 1_{\{z\geq 0\}} \qquad \nu_2(z) = \rho_2 e^{\rho_2 z} 1_{\{z\leq 0\}}. \tag{42}$$

In this case, first and second moments of $O_1$ and $O_2$ are respectively equal to $\mu_1 = \frac{1}{\rho_1}$, $\mu_2 = -\frac{1}{\rho_2}$ and to $\eta_i = \frac{2}{(\rho_i)^2}$. The moment generating functions are given by $\psi_1(z) = \frac{\rho_1}{\rho_1 - z}$ for $z < \rho_1$ and $\psi_2(z) = \frac{\rho_2}{\rho_2+z}$ for $z > -\rho_2$. In this particular, we have the following interesting corollary:

**Corollary 3.3.** *The distribution of orders are exponential under $P$ and $Q$ and the densities, noted $\nu_i^Q(z)$ under $Q$, are defined by parameters:*

$$\begin{aligned} \rho_1^Q &= \rho_1 - (\delta_{1,1}a_1 + \delta_{2,1}a_2 + \theta_1) \\ \rho_2^Q &= \rho_2 + (a_2\delta_{2,2} + a_1\delta_{1,2} + \theta_2) \end{aligned}$$

*Proof.* If we denote $\beta_1 = \delta_{1,1}a_1 + \delta_{2,1}a_2 + \theta_1$ , by construction the moment generating function of sell orders, under the risk neutral measure is provided by the following ratio:

$$\begin{aligned} \psi_1^Q(z) &= \frac{\psi_1(z+\beta_1)}{\psi_1(\beta_1)} \\ &= \frac{\rho_1}{\rho_1 - z - \beta_1}\frac{\rho_1 - \beta_1}{\rho_1} \end{aligned}$$

and we conclude that sell orders are also exponential under $Q$. The same reasoning holds for ask orders. □

Under $Q^{\theta_1,\theta_2}$, dynamics of intensities are preserved as shown in the next corollary.

**Corollary 3.4.** *Intensities of counting processes $N_t^{1,Q}$ and $N_t^{2,Q}$ are Hawkes processes having the same structure under $Q$ as these under the real measure $P$*

$$d\lambda_t^{i,Q} = \kappa_i(c_i^Q - \lambda_t^{i,Q})dt + \delta_{i,1}^Q dL_t^{1,Q} + \delta_{i,2}^Q dL_t^{2,Q} \quad i = 1, 2. \tag{43}$$

*where the parameters defining the process under $Q$ are:*

$$\begin{aligned} c_1^Q &= c_1\psi_1(a_1\delta_{1,1} + a_2\delta_{2,1} + \theta_1) \\ c_2^Q &= c_2\psi_2(a_2\delta_{2,2} + a_1\delta_{1,2} + \theta_2) \\ \delta_{1,j}^Q &= \delta_{1,j}\psi_1(a_1\delta_{1,1} + a_2\delta_{2,1} + \theta_1) \quad j = 1, 2 \\ \delta_{2,j}^Q &= \delta_{2,j}\psi_2(a_2\delta_{2,2} + a_1\delta_{1,2} + \theta_2) \quad j = 1, 2 \end{aligned}$$

This corollary is proved by combining equations (33) and (36). If markets participants adopt an equivalent exponential affine measure for the risk neutral one, the price of a zero coupon bond is equal to the expected discount factor, under this risk neutral measure. The price is denoted by:

$$\begin{aligned} P(t, T, \lambda_t^1, J_t^1, \lambda_t^2, J_t^2) &= \mathbb{E}^Q\left(e^{-\int_t^T r_s ds} \,|\, \mathcal{F}_t\right) \\ &= e^{-\int_t^T \varphi(s)ds}\mathbb{E}^Q\left(e^{-\int_t^T X_s ds} \,|\, \mathcal{F}_t\right). \end{aligned} \tag{44}$$

and the expectation in the left term of the bond price is provided in the following corollary, that is proved by combining the proposition 2.4 with the corollary 3.4.

**Corollary 3.5.**

$$\mathbb{E}^Q\left(e^{-\int_t^T X_s ds}\,|\,\mathcal{F}_t\right) \;=\; \exp\left(-X_t(T-t)+A(t,T)+\begin{pmatrix} B_1(t,T)\\ B_2(t,T)\end{pmatrix}^\top\begin{pmatrix}\lambda_t^{1Q}\\ \lambda_t^{2Q}\end{pmatrix}\right) \tag{45}$$

*where $A(t,T)$ , $B_1(t,T)$ and $B_2(t,T)$ are solutions of a system of ODEs :*

$$\begin{cases} \frac{\partial}{\partial t}B_1(t,T)=\kappa_1 B_1(t,T)-\left[\psi_1^Q\left(B_1(t,T)\delta_{1,1}^Q-\alpha_1(T-t)+B_2(t,T)\delta_{2,1}^Q\right)-1\right]\\ \frac{\partial}{\partial t}B_2(t,T)=\kappa_2 B_2(t,T)-\left[\psi_2^Q\left(B_1(t,T)\delta_{1,2}^Q+\alpha_2(T-t)+B_2(t,T)\delta_{2,2}^Q\right)-1\right]\\ \frac{\partial}{\partial t}A(t,T)=-\kappa_1 c_1^Q B_1(t,T)-\kappa_2 c_2^Q B_2(t,T)\end{cases} \tag{46}$$

*with the terminal conditions $A(T,T)=0$ , $B_1(T,T)=0$, $B_2(T,T)=0$.*

The dynamics of bond prices depends upon the random measures of jump processes, noted $L^{1Q}(dt,dz)$ and $L^{2Q}(dt,dz)$ and such that:

$$L_t^{kQ} \;=\; \int_0^\infty\int_{-\infty}^\infty L^{kQ}(dt,dz) \quad k=1,2.$$

Furthermore, the expectation of these measures are equal to $\mathbb{E}(L^{kQ}(dt,dz)|\mathcal{F}_t)=\lambda_t^{kQ}\nu_k(z)\,dz\,dt$, for $k=1,2$. The next corollary details the infinitesimal dynamics of bond prices:

**Corollary 3.6.** *Bond prices,$P(t,T,\lambda_t^{1Q},J_t^{1Q},\lambda_t^{2Q},J_t^{2Q})$, are ruled by the following SDE:*

$$\begin{aligned} dP \;=\;& P\,r_t\,dt-\lambda_t^{1Q}P\left[\psi_1^Q\left(B_1(t,T)\delta_{1,1}^Q-\alpha_1(T-t)+B_2(t,T)\delta_{2,1}^Q\right)-1\right]dt \\ &+P\int_{-\infty}^{+\infty}\exp\left(\left(B_1(t,T)\delta_{1,1}^Q-\alpha_1(T-t)+B_1(t,T)\delta_{2,1}^Q\right)z\right)-1\,L^{1Q}(dt,dz)\\ &-\lambda_t^{2Q}P\left[\psi_2^Q\left(B_1(t,T)\delta_{1,2}^Q+\alpha_2(T-t)+B_2(t,T)\delta_{2,2}^Q\right)-1\right]dt\\ &+P\int_{-\infty}^{+\infty}\exp\left(\left(B_1(t,T)\delta_{1,2}^Q+\alpha_2(T-t)+B_2(t,T)\delta_{2,2}^Q\right)z\right)-1\,L^{2Q}(dt,dz)\end{aligned} \tag{47}$$

*where $L^{1Q}(dt,dz)$ and $L^{2Q}(dt,dz)$ are the random measures of jump processes.*

*Proof.* According to the Itô's lemma for semi-martingales, $P(t,T,\lambda_t^{1Q},J_t^{1Q},\lambda_t^{2Q},J_t^{2Q})$ is such that

$$\begin{aligned} dP =\;& P_t+\kappa_1(c_1^Q-\lambda_t^{1Q})P_{\lambda^1}dt+\kappa_2(c_2^Q-\lambda_t^{2Q})P_{\lambda^2}dt \\ &+\int_{-\infty}^{+\infty}P(t,\lambda_t^{1Q}+\delta_{1,1}^Q z,J_t^{1Q}+(z,1)^\top,\lambda_t^{2Q}+\delta_{2,1}^Q z,J_t^{2Q})-P(t,\lambda_t^{1Q},J_t^{1Q},\lambda_t^{2Q},J_t^{2Q})\,L^{1Q}(dt,dz)\\ &+\int_{-\infty}^{+\infty}P(t,\lambda_t^{1Q}+\delta_{1,2}^Q z,J_t^{1Q},\lambda_t^{2Q}+\delta_{2,2}^Q z,J_t^{2Q}+(z,1)^\top)-P(t,\lambda_t^{1Q},J_t^{1Q},\lambda_t^{2Q},J_t^{2Q})\,L^{2Q}(dt,dz),\end{aligned} \tag{48}$$

where partial derivatives are obtained from equations (45) and (46) □

From the last corollary, we infer that the instantaneous growth rate for the bond price is well equal to the short-term rate, $\mathbb{E}\left(\frac{dP}{P}|\mathcal{F}_t\right)=r_t dt$, as the sum of all other terms in equation (47) is a martingale.

## Pricing of options.

This section illustrates how the model is used for the pricing of interest rate derivatives, under a forward measure. The yield of maturity $T-S$, at time $T$ is denoted by $Y(T,S)$ and is defined by:

$$\begin{aligned} Y(T,S) &:= -\frac{1}{S-T}\log P(T,S) \qquad (49)\\ &= X_T+\frac{1}{S-T}\left(\int_T^S \varphi(s)ds - A(T,S)\right)-\frac{1}{S-T}\begin{pmatrix} B_1(T,S)\\ B_2(T,S)\end{pmatrix}^\top \begin{pmatrix}\lambda_T^{1Q}\\ \lambda_T^{2Q}\end{pmatrix}\end{aligned}$$

On another hand, the payoff paid at time $S\geq T$ by an European option written on $Y(T,S)$ is denoted by $V(Y(T,S))$. Examples of such instruments are: caplets ($V(Y(T,S))=N(S-T)[Y(T,S)-k]_+$), floorlets ( $V(Y(T,S))=N(S-T)N[k-Y(T,S)]_+$ ) or options of zero coupon bonds ($V(Y(T,S))=N[\exp(-Y(T,S)(S-T))-k]_+$), where $N$ and $k$ are respectively the principal and the strike. The option price is the expectation of this discounted payoff under the risk neutral measure:

$$f(t,r_t,\lambda_t) \;=\; \mathbb{E}^Q\left(e^{-\int_t^S r_s ds}V(Y(T,S))\,|\,\mathcal{F}_t\right). \qquad (50)$$

As recommended by Brigo and Mercurio (2007), it is better to evaluate this last expression under the $S-$forward measure. This avoids numerical inaccuracies related to the approximation of $\exp\left(-\int_t^S r_s ds\right)$, because the discount factor is drawn out of the equation (50), under the forward measure. If the market admits at least one risk neutral measure $Q$, an equivalent probability measures to $Q$ is defined by the technique of changes of numeraire. The $S$-forward measure has as numeraire, the zero coupon bond of maturity $S$. Under this measure, the price of any financial assets, divided by the numeraire $P(t,S)$, is a martingale and the price of the derivative is equal to:

$$\begin{aligned}\mathbb{E}^Q\left(e^{-\int_t^S r_s ds}V(Y(T,S))\,|\,\mathcal{F}_t\right) &= P(t,S)\mathbb{E}^S\left(V(Y(T,S))\,|\,\mathcal{F}_t\right)\\ &= P(t,S)\int_0^{+\infty} V(y)f_{Y(T,S)}(y)dy\end{aligned}$$

where $f_{Y(T,S)}(y)$ is the density of $Y(T,S)$ under the forward measure. If $B(t)$ points out here the market value of a cash account, $B_t=e^{\int_0^t r_s ds}$, the Radon Nykodym derivative defining the $S$-forward measure, is equal to:

$$\frac{dF^S}{dQ} \;=\; \frac{1}{B_S}\frac{B_0}{P(0,S)}=\left(e^{\int_0^S r_s ds}\mathbb{E}^Q\left(e^{-\int_0^S r_s ds}|\mathcal{F}_0\right)\right)^{-1}$$

To calculate the expected payoff under $F^S$, the easiest approach consists to approximate the probability density function of $Y(T,S)$ by a Discrete Fourier Transform. To perform a such calculation, the moment generating function of the yield is needed:

**Corollary 3.7.** *The moment generating function of $Y(T,S)$ at time $t\leq T$ under the forward measure $F^S$, denoted by $\varphi^{t,S}(w)$, is given by:*

$$\begin{aligned}\varphi^{t,S}(w) &= \mathbb{E}^S\left(e^{wY(T,S)}\,|\,\mathcal{F}_t\right)\\ &= \exp\left(\left(\frac{w}{S-T}\right)\int_T^S \varphi(s)ds + wX_t\right)\\ &\times \exp\left(A^T(t,T)-A^S(t,S)+\left(\begin{pmatrix} B_1^T(t,T)-B_1^S(t,S)\\ B_2^T(t,T)-B_2^S(t,S)\end{pmatrix}^\top\begin{pmatrix}\lambda_T^{1Q}\\ \lambda_T^{2Q}\end{pmatrix}\right)\right)\end{aligned}$$

*where $A^S(t,S)$ , $B_1^S(t,S)$ and $B_2^S(t,S)$ are solutions of the system of ODEs (46) with a maturity $S$ and where $A^T(t,T)$ , $B_1^T(t,T)$ and $B_2^T(t,T)$ are solutions of the following system of ODEs :*

$$\begin{cases} \frac{\partial}{\partial t}B_1^T(t,T) = \kappa_1 B_1^T(t,T) \\ \qquad\qquad - \left[\psi_1\left(B_1^T(t,T)\delta_{1,1}^Q + (w-(S-t))\,\alpha_1 + B_2^T(t,T)\delta_{2,1}^Q\right) - 1\right] \\ \frac{\partial}{\partial t}B_2^T(t,T) = \kappa_2 B_2^T(t,T) \\ \qquad\qquad - \left[\psi_2\left(B_1^T(t,T)\delta_{1,2}^Q - (w-(S-t))\,\alpha_2 + B_2^T(t,T)\delta_{2,2}^Q\right) - 1\right] \\ \frac{\partial}{\partial t}A^T(t,T) = -\kappa_1 c_1^Q B_1^T(t,T) - \kappa_2 c_2^Q B_2^T(t,T) \end{cases} \tag{51}$$

*with the terminal conditions $A^T(T,T) = \left(1-\frac{w}{S-T}\right)A^S(T,S)$ , $B_1^T(T,T) = \left(1-\frac{w}{S-T}\right)B_1^S(T,S)$, $B_2^T(T,T) = \left(1-\frac{w}{S-T}\right)B_2^S(T,S)$.*

*Proof.* By definition of the forward measure and using the fact that $\mathcal{F}_t \subset \mathcal{F}_T$, the Laplace transform of $Y(T,S)$ is given by:

$$\begin{aligned} \mathbb{E}^S\left(e^{wY(T,S)} \mid \mathcal{F}_t\right) &= \frac{\mathbb{E}^Q\left(\left(e^{\int_0^S r_s ds}\mathbb{E}^Q\left(e^{-\int_0^S r_s ds}|\mathcal{F}_0\right)\right)^{-1} e^{wY(T,S)} \mid \mathcal{F}_t\right)}{e^{-\int_0^t r_s ds}\mathbb{E}^Q\left(e^{-\int_t^S r_s ds}|\mathcal{F}_t\right)\left(\mathbb{E}^Q\left(e^{-\int_0^S r_s ds}|\mathcal{F}_0\right)\right)^{-1}} \\ &= \frac{\mathbb{E}^Q\left(e^{-\int_t^T r_s ds}\mathbb{E}^Q\left(e^{-\int_T^S r_s ds + wY(T,S)} \mid \mathcal{F}_T\right) \mid \mathcal{F}_t\right)}{\mathbb{E}^Q\left(e^{-\int_t^S r_s ds}|\mathcal{F}_t\right)}. \end{aligned}$$

The $\mathcal{F}_T$ conditional expectation in this last equation, is also equal to

$$\mathbb{E}^Q\left(e^{-\int_T^S r_s ds + wY(T,S)} \mid \mathcal{F}_T\right) = e^{wY(T,S)}\mathbb{E}^Q\left(e^{-\int_T^S r_s ds} \mid \mathcal{F}_T\right),$$

and, according the corollary 3.5, we have that:

$$\begin{aligned} \mathbb{E}^Q\left(e^{-\int_T^S r_s ds + wY(T,S)} \mid \mathcal{F}_T\right) &= \exp\left(\left(\frac{w}{S-T}-1\right)\left(\int_T^S \varphi(s)ds - A^S(T,S)\right)\right) \\ &\times \exp\left(\left(\frac{w}{S-T}-1\right)\left(X_T(S-T) - \begin{pmatrix} B_1^S(T,S) \\ B_2^S(T,S) \end{pmatrix}^\top \begin{pmatrix} \lambda_T^{1Q} \\ \lambda_T^{2Q} \end{pmatrix}\right)\right) \end{aligned}$$

and

$$\mathbb{E}^Q\left(e^{-\int_t^S r_s ds} \mid \mathcal{F}_t\right) = \exp\left(-X_t(S-t) - \int_t^S \varphi(s)ds + A^S(t,S) + \begin{pmatrix} B_1^S(t,S) \\ B_2^S(t,S) \end{pmatrix}^\top \begin{pmatrix} \lambda_t^{1Q} \\ \lambda_t^{2Q} \end{pmatrix}\right),$$

Using the proposition (2.4) allows us to conclude. □

The next result introduces the discretization framework to build the density of $Y(T,S)$, under the forward measure. Note that it is possible to use the same algorithm to approach the distribution of $r_t$ under the real and risk neutral measure.

**Proposition 3.8.** *Let $M$ be the number of steps used in the Discrete Fourier Transform (DFT) and $\Delta_y = \frac{2y_{max}}{M-1}$ be this step of discretization. Let us denote $\Delta_z = \frac{2\pi}{M\,\Delta_y}$ and*

$$z_j = (j-1)\Delta_z,$$

*for* $j = 1...M$. *The values of* $f_{Y(T,S)}(.)$ *at points* $y_k = -\frac{M}{2}\Delta_y + (k-1)\Delta_y$ *are approached by the sum:*

$$f_{Y(T,S)}(y_k) \approx \frac{2}{M\,\Delta_y} Re\left(\sum_{j=1}^{M} \delta_j \varphi^{t,S}\left(i\,z_j, r_t, \lambda_t\right)(-1)^{j-1} e^{-i\frac{2\pi}{M}(j-1)(k-1)}\right). \tag{52}$$

*where* $\delta_j = \frac{1}{2}1_{\{j=1\}} + 1_{\{j\neq 1\}}$.

*Proof.* The density of $Y(T,S)$ is retrieved by calculating the Fourier transform of $\varphi^{t,T}(iz)$ as follows:

$$\begin{aligned} f_{Y(T,S)}(y_k) &= \frac{1}{2\pi}\mathcal{F}[\varphi^{t,S}(iz)](y) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi^{t,S}(iz)\,e^{-i\,y_k\,z}dz \\ &= \frac{1}{\pi}Re(\int_{0}^{+\infty} \varphi^{t,S}(iz)e^{-i\,y_k\,z}dz) \end{aligned}$$

where the last equality comes from the fact that $\varphi^{t,S}(z)$ and $\varphi^{t,T}(-z)$ are complex conjugate. At points $y_k = -\frac{M}{2}\Delta_y + (k-1)\Delta_y$, this last integral is approached with the trapezoid rule

$$\int_a^b h(z)dz = \frac{h(a)+h(b)}{2}\Delta_z + \sum_{k=1}^{M-1} h(a+k\,\Delta_z)\Delta_z$$

and leads to the following estimate for $f_{Y(T,S)}(y_k)$:

$$\begin{aligned} f_{Y(T,S)}(y_k) &\approx \frac{1}{\pi}Re\left(\sum_{j=1}^{M}\delta_j\varphi^{t,S}(iz_j)e^{-i\,y_k z_j}\Delta_z\right) \\ &\approx \frac{1}{\pi}Re\left(\sum_{j=1}^{M}\delta_j\varphi^{t,S}(iz_j)(-1)^{j-1}e^{-i\frac{2\pi}{M}(j-1)(k-1)}\Delta_z\right) \end{aligned}$$

□

Once that the density of $Y(T,S)$ is obtained by the discrete Fourier transform, the option price is approached by a weighted sum of payoffs:

$$\mathbb{E}^Q\left(e^{-\int_t^T r_s ds}V(Y(T,S))\,|\,\mathcal{F}_t\right) = P(t,T)\sum_{k=1}^{M+1} V(y_k)f_{Y(T,S)}(y_k)\Delta_y\,.$$

The feasibility of this method is illustrated for caplets, in the numerical application.

# 4 Calibration and numerical applications.

To demonstrate that the model is adequate for interest rate modeling, we first perform an econometric calibration. The data set used is made up zero coupon rates, bootstrapped from daily Euro swap rates (bid-ask average), observed over ten years (3/05/2004 to 30/12/2014). Swaps are liquid instruments, and their rates are representative of yields of AA corporate bonds. The maturities of considered swaps are running from 1 to 10 years, 12 15 and 20 years. The Bloomberg tickers are EUSA1 to EUSA10, EUSA12, EUSA15 and EUSA20 and the field is PX_LAST. Figure (1) provides a three-dimensional plot of zero coupon rates. The large amount of temporal variation in the level is visually apparent. The flattening of the curvature during the 2008 crisis, is also clearly visible. Table 1 shows descriptive statistics for swap rates. The typical curve is upward sloping and long term rates are less volatile and more persistent than short term rates (in the sense that their long term auto-correlation is higher).

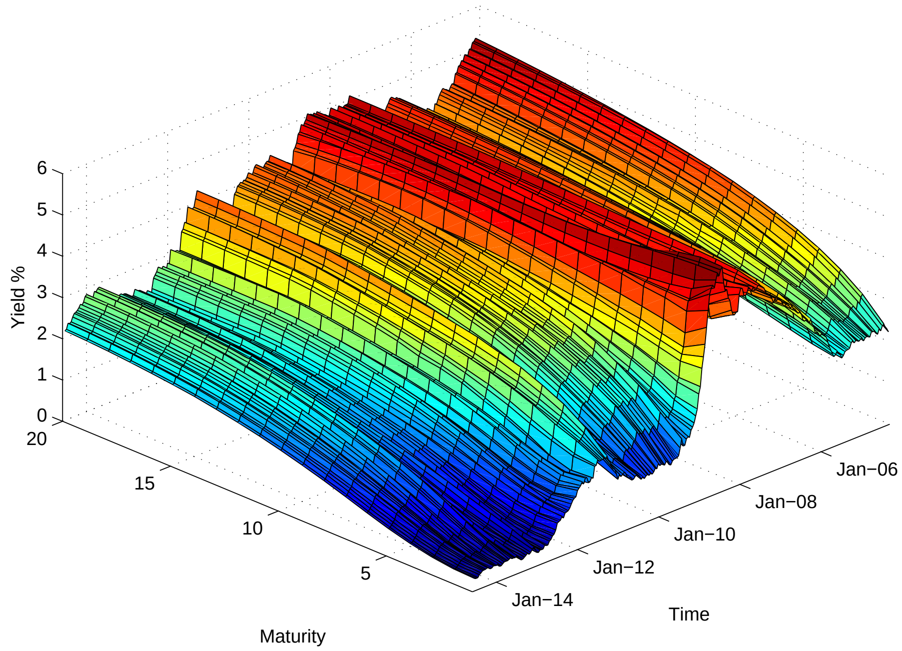


Figure 1: This graph shows the evolution of zero coupon rates, bootstrapped from swap curves, over a ten year period (3/05/2004 to 25/7/2014).

| Maturity | Mean | Std. dev. | Minimum | Maximum | $\rho(175d)$ | $\rho(250d)$ |
|---|---|---|---|---|---|---|
| 1 | 0.0217 | 0.0147 | 0.0029 | 0.0545 | 0.7929 | 0.6759 |
| 2 | 0.0229 | 0.0140 | 0.0031 | 0.0548 | 0.8114 | 0.7315 |
| 3 | 0.0245 | 0.0134 | 0.0038 | 0.0540 | 0.8222 | 0.7614 |
| 4 | 0.0261 | 0.0127 | 0.0049 | 0.0528 | 0.8265 | 0.7746 |
| 5 | 0.0277 | 0.0121 | 0.0062 | 0.0519 | 0.8271 | 0.7800 |
| 6 | 0.0292 | 0.0115 | 0.0077 | 0.0513 | 0.8273 | 0.7837 |
| 7 | 0.0305 | 0.0110 | 0.0093 | 0.0510 | 0.8265 | 0.7852 |
| 8 | 0.0317 | 0.0105 | 0.0109 | 0.0508 | 0.8252 | 0.7855 |
| 9 | 0.0327 | 0.0102 | 0.0124 | 0.0507 | 0.8232 | 0.7844 |
| 10 | 0.0337 | 0.0099 | 0.0138 | 0.0509 | 0.8197 | 0.7813 |
| 12 | 0.0353 | 0.0094 | 0.0161 | 0.0513 | 0.8114 | 0.7728 |
| 15 | 0.0370 | 0.0090 | 0.0187 | 0.0516 | 0.7990 | 0.7596 |
| 20 | 0.0380 | 0.0089 | 0.0187 | 0.0526 | 0.7904 | 0.7507 |

Table 1: Descriptive statistics, zero coupon rates bootstrapped from swap curves from 3/05/2004 to 25/7/2014. The two last columns contains sample auto-correlation at displacement of 175 days and 250 days of trading.

The parameters that define the dynamics of the short term rate under the real measure, are fitted to the time series of one year swap rates (presented in the first subplot of figure 2).

Positive and negative jumps are both assumed to be exponential random variables, with means $\frac{1}{\rho_1}$ and $-\frac{1}{\rho_2}$. For this choice of distributions, parameters $\alpha_1$ and $\alpha_2$ are redundant and set to one. The function of time $\varphi(t)$ is assumed constant and equal to the one year swap rate, on the 3/05/2004. In practice, $\varphi(t)$ is used to perfectly duplicates the most recent yield curve and to exclude any possibilities of arbitrage in the pricing of interest rate derivatives. As our purpose is here econometric, setting $\varphi(t)$ to a constant is not penalizing. Positive (1360 observations) and negative (1460 observations) variations of the one year rate are respectively assimilated to an increase of supply and increase of demand, the parameters $\rho_1$ and $\rho_2$ are adjusted by matching the first moment. The intensities $\lambda_t^1$ and $\lambda_t^2$ are fitted separately by direct log-likelihood maximization procedures. If daily variations of interest rates are denoted by $\Delta r_i = r_{t_i} - r_{t_{i-1}}$ for $i = 1$ to $n = 2820$ observations and $\Delta_t$ is the length of the time interval, the following two optimization problems are solved numerically to find an estimate of parameters:

$$\begin{cases} (\kappa_1, c_1, \delta_{1,1}, \delta_{1,2}) & = \arg\max \sum_{i=1}^{n} \log\left(\lambda_{t_i}^1 \Delta_t 1_{\{\Delta r_i > 0\}} + (1-\lambda_{t_i}^1 \Delta_t) 1_{\{\Delta r_i \leq 0\}}\right) \\ (\kappa_2, c_2, \delta_{2,1}, \delta_{2,2}) & = \arg\max \sum_{i=1}^{n} \log\left(\lambda_{t_i}^2 \Delta_t 1_{\{\Delta r_i < 0\}} + (1-\lambda_{t_i}^2 \Delta_t) 1_{\{\Delta r_i \geq 0\}}\right) \end{cases}$$

where the intensity of the arrival of jumps is discretized as follows :

$$\lambda_{t_i}^k = \lambda_{t_{i-1}}^k + \kappa_k (c_k - \lambda_{t_{i-1}}) \Delta_t + \delta_{k,1} |\Delta r_i| 1_{\Delta r_i \geq 0} + \delta_{k,2} |\Delta r_i| 1_{\Delta r_i \geq 0} \quad k = 1, 2\ ,\ i = 1, ..., n.$$

The results of the calibration procedure are presented in table 2. The speeds of mean reversion for the intensities of supply and demand are close and around 3.90 . The $\delta_{1,1}$ and $\delta_{2,2}$ measure the level of self excitation and are positive. This confirms the presence of marginal clustering effects in the frequency of orders. The marginal effect of the demand on supply, such as measured by $\delta_{1,2}$, is negative. This means that an upward shift in demand decreases the frequency of supply orders. But as $|\delta_{1,2}|\frac{1}{\rho_2} < \delta_{1,1}\frac{1}{\rho_1}$, this effect is less significant on average than the self excitation. $\delta_{2,1}$ is also negative and then an increase of supply decreases the frequency of demand orders. Compared to the self excitation, this effect is predominant on average as $|\delta_{2,1}|\frac{1}{\rho_1} > \delta_{2,2}\frac{1}{\rho_2}$. This means that the supply drives the demand for bonds rather than the opposite.

| Parameter | Value | std. err. | Parameter | Value | std. err. |
|---|---|---|---|---|---|
| $\kappa_1$ | 3.86 | 0.0187 | $\delta_{1,1}$ | 5619.40 | 10.3072 |
| $\kappa_2$ | 3.93 | 0.0196 | $\delta_{1,2}$ | -4681.94 | 8.1994 |
| $c_1$ | 120.00 | 0.0643 | $\delta_{2,1}$ | -5486.30 | 10.1354 |
| $c_2$ | 134.63 | 0.0621 | $\delta_{2,2}$ | 4608.52 | 7.5390 |
| $\alpha_1$ | 1 | | $\alpha_2$ | 1 | |
| $\rho_1$ | 5653.08 | 6.0047 | $\rho_2$ | 5576.20 | 6.0875 |

Table 2: This table contains the parameters defining $dX_t$ under the real measure and their standard errors.

The exact calculation of the total log-likelihood would require to estimate 2820 pdf by DFT given that the probability density function of $r_t$ depends on $\lambda_t^1$ ,$\lambda_t^2$ and does not admit a closed form expression. As this is computationally too intensive, the total log-likelihood of the model is instead approached by the following expression:

$$\begin{aligned} L &= \sum_{i=1}^{n} \ln\left[1_{\Delta r_i \geq 0}\left(\lambda_{t_i}^1 \Delta t + \left(1 - \left(\lambda_{t_i}^2 \Delta t\right)\right)\right) \nu_1(\Delta r_i) + \right. \\ & \left. + 1_{\Delta r_i < 0}\left(\lambda_{t_i}^2 \Delta t + \left(1 - \left(\lambda_{t_i}^1 \Delta t\right)\right)\right) \nu_2(\Delta r_i)\right] , \end{aligned}$$

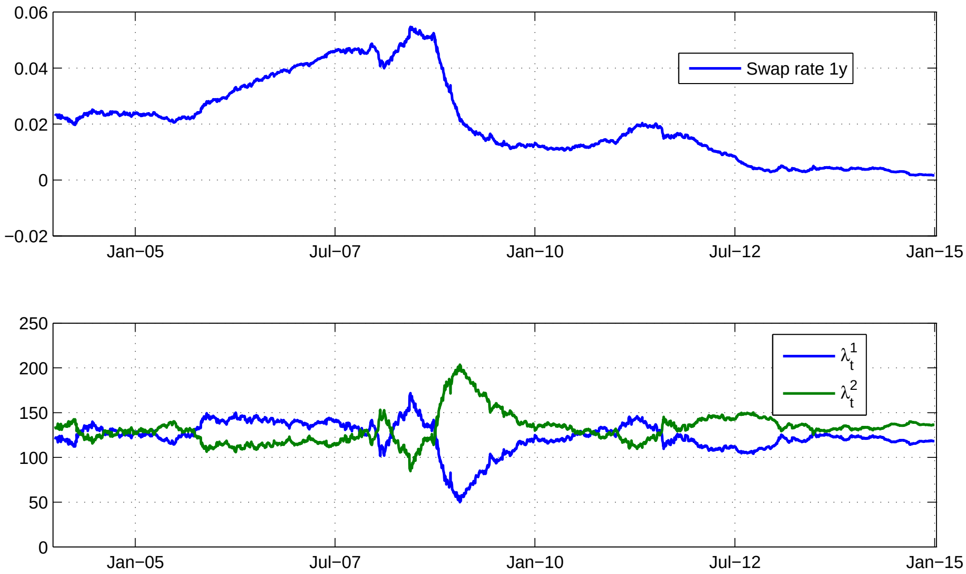


Figure 2: The first subplot presents the history of the one year swap rate, to which the model is fitted. The second graph shows the sample paths of $\lambda_t^1$ and $\lambda_t^2$ obtained by log likelihood maximization.

and is equal to 23 455 whereas the AIC (Akaike Information criterion) is -46 891. To benchmark the model, we fit a Vasicek model (1977) to the same data set. The log-likelihood for this model is equal to 19 078 and its AIC is -38 150. From the comparison of AIC, we deduce then that our model outperforms the Vasicek model (using the BIC leads to the same conclusion).

Since the inception of the euro in 1999 and the resulting elimination of exchange-rate risk, swap rates reflect the fluctuations of compensations demanded by AA rated financial institutions for holding mainly liquidity risks. Liquidity risk arises from the potential difficulty to find a counterpart to close a trade relatively quickly. Aît-Sahalia et al. (2014 a) measure stocks market stress with Hawkes jumps intensities and mention that they reflect market conditions at the time. In a similar way, anticipating poor liquidity conditions is possible through the analysis of filtered intensities, as shown in the second graph of figure 2. The frequency of supply orders $\lambda_t^1$ is most of the time around the frequency of demand orders $\lambda_t^2$, and they have symmetric patterns of evolution. When $\lambda_t^2$ is far below $\lambda_t^1$ , bid orders are not enough frequent compared to ask orders and the market is threatened by a liquidity shortfall. This scenario, happens from the 14/08/2007 to the 09/12/2008, the period that corresponds to the credit crunch crisis.

The econometric calibration is based on historical data and parameters obtained by a such approach define the dynamics of $r_t$ under the real measure of probability $P$. To appraise parameters under the risk neutral measure, we find the risk premiums $\theta_1$ and $\theta_2$ defining an equivalent measure by the equation (28) that minimizes the sum of spreads between model-based and observed yields. In practice, it is not relevant to assume that these premiums are constant over a period of 10 years, as they are directly related to the level of risk aversion in financial markets. We have then computed the risk premiums at regular interval of five days of trading. The first subplot of figure 3 shows their evolution: $\theta_1$ and $\theta_2$ are respectively positive and negative and nearly symmetric till 2013. The second subplot presents the evolution of $\psi_1(a_1\delta_{1,1} + a_2\delta_{2,1} + \theta_1)$ and $\psi_2(a_2\delta_{2,2} + a_1\delta_{1,2} + \theta_2)$ that multiply parameters $c_1$, $c_2$ and $\delta_{i,j}$ under $Q$, as stated in the corollary 3.4. This graph reveals that parameters driving $\lambda_t^1$ (resp. $\lambda_t^2$) are always increased (resp.

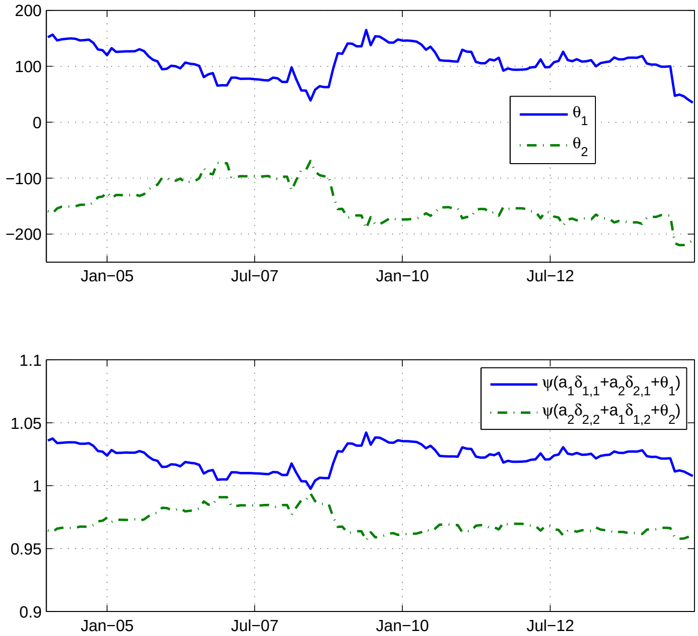


Figure 3: The first subplot exhibits the history of risk premiums, appraised at regular interval of five days. The second graph shows coefficients $\psi_1(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)$ and $\psi_2(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)$ that multiply the parameters defining $\lambda_t^1$ and $\lambda_t^2$ under the risk neutral measure.

decreased) under $Q$. And the steepness of the yield curve is directly related to the distance between $\psi_1(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)$ and $\psi_2(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)$. Around the credit crunch, yield curves are indeed nearly flat and $\psi_1(a_1\delta_{1,1}+a_2\delta_{2,1}+\theta_1)$ and $\psi_2(a_2\delta_{2,2}+a_1\delta_{1,2}+\theta_2)$ are very close to one.

| | Value under Q | Best fit | | Value under Q |
|---|---|---|---|---|
| $\kappa_1$ | 3.863 | 1.4463 | $\delta_{1,1}^Q$ | 5662.25 |
| $\kappa_2$ | 3.928 | 7.6587 | $\delta_{1,2}^Q$ | -4717.21 |
| $c_1^Q$ | 120.91 | 112.54 | $\delta_{2,1}^Q$ | -5270.10 |
| $c_2^Q$ | 129.33 | 128.78 | $\delta_{2,2}^Q$ | 4426.65 |
| $\lambda_0^{1,Q}$ | 119.08 | | $\lambda_0^{1,Q}$ | 131.08 |
| $r_0$ | 0.0018 | | | |
| $\rho_1^Q$ | 5609.62 | | $\rho_2^Q$ | 5804.41 |

Table 3: This table presents the parameters defining the dynamics of $r_t$ under the risk neutral measure, on the 28/11/2014. They are obtained by adjusting historical parameters according to corollary 3.4. The values of $\kappa_1$, $\kappa_2$, $c_1^Q$ and $c_2^Q$ in the column "Best fit" are these for which the model replicates accurately the yield curve. But they cannot be reconciled with historical parameters by an affine change of measure.

Table 3 shows the parameters under $Q$ on the 28/11/2014. The first subplot of figure 4 compares the yields produced by the model for these parameters (line labeled "Model $Q$") with the observed ones. If we replace $\kappa_1$, $\kappa_2$, $c_1^Q$ and $c_2^Q$ by values in the column "Best fit" of table 3, the model replicates perfectly the market data. This is illustrated in the first subplot of figure 4 by the curve labeled "Best fit". These parameters cannot be reconciled anymore with historical parameters, through an affine change of measure but does not appear irrelevant. We use them later to analyze the sensitivity of the model to each of its parameters.

The five last subplots of the figure 4 shows the marginal effect of each parameter on the slope of the yield curve produced by the model. Increasing $c_1$, $\delta_{1,1}$ or $\delta_{1,2}$ raises the frequency of positive variations of the interest rate, and then the steepness of the curve. Increasing $c_2$, $\delta_{2,2}$ or $\delta_{2,1}$ steps up the frequency of negative variations of the short term rate and flatten the yield curve. Using higher speeds of mean reversion, $\kappa_1$and $\kappa_2$, have the same effect on the curve. As the average size of orders are inversely proportional to $\rho_1$ and $\rho_2$, increasing these parameters is equivalent to decrease the average amplitude of variations of the interest rate. Then higher $\rho_1$ or $\rho_2$ respectively lowers or raises the steepness of the curve.

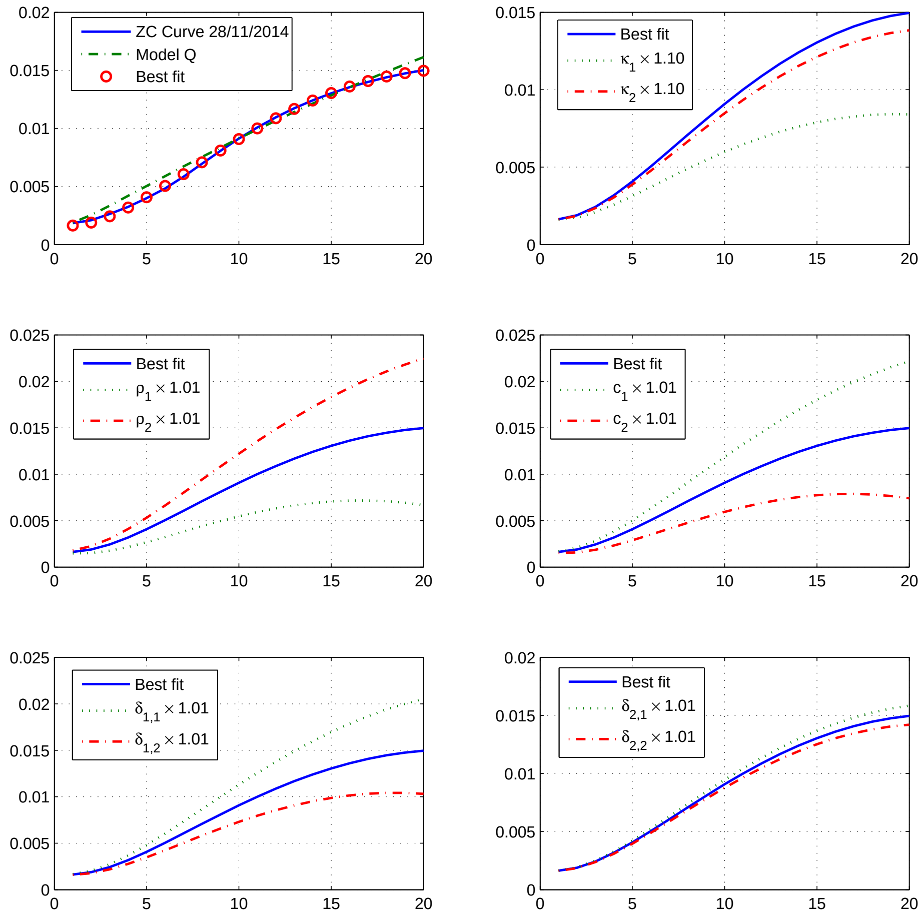


Figure 4: The first subplot shows the zero coupon yield curve, on the 28/11/2014 and the curves built with the model and sets of parameters of table 3. The five last subplots illustrate the sensitivity of the yield curve produced by the model to changes of parameters.

The figure 5 presents simulated sample paths for $r_t$, $\lambda_t^1$ and $\lambda_t^2$ and their mean calculated under $Q$, with propositions 2.1 and 2.2. Simulated paths depict periods of decline, sharp increase and stability, that are comparable to real ones shown in figure 2. We also observe negative short term rates in two scenarios during the first five years, as the expected short term rate is close to 0%. In fact, the number of scenarios in which negative rates are generated, directly depends on parameters of mutual excitations $\delta_{1,2}$ and $\delta_{2,1}$. The higher is $\delta_{1,2}$, the higher is the probability of observing an upward jump following a downward variation of interest rates and the lower is the probability of observing negative short term rates. This point is emphasized by the first subplot of figure 6, that presents the probability density function (computed by DFT) of the forward yield, $Y(2,3)$ as defined by equation (49), for different levels of cross excitations. We see that setting $\delta_{1,2}$ to zero is enough to exclude negative forward yields.

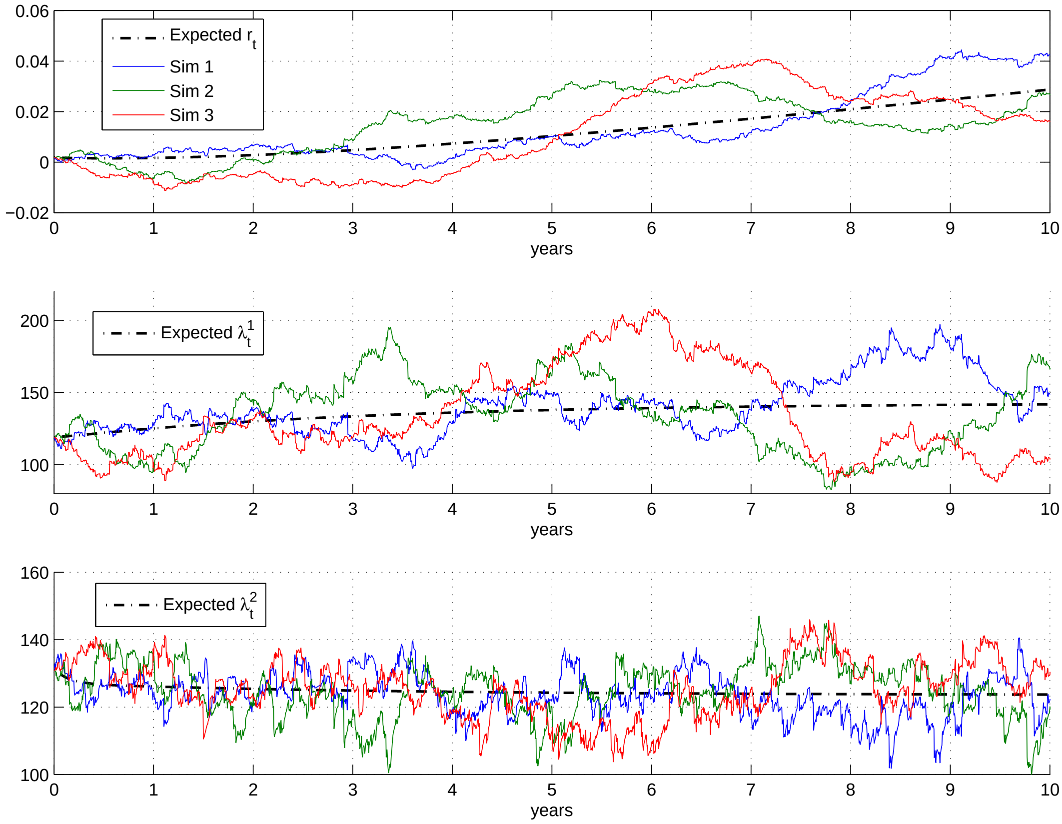


Figure 5: This graph displays three simulated sample paths of $r_t$ , and the intensity of arrivals of sell $(\lambda_t^1)$ / buy $(\lambda_t^2)$ orders. The period is 10 years.

The five last subplots of figure 6 show selected curves of implied volatilities, for a set of 2 years caplets, with a 1 year tenor. Prices are obtained by a Fourier transform with $M = 2^{12}$ steps of discretization and $y_{max} = 0.10$. Implied volatilities are next obtained by inverting the Black & Scholes formula for caplets. The purpose of these graphs is to illustrate the sensitivity of implied volatilities to a change of key parameters defining the model. Parameters are these fitting market on the 28/11/2014. Increasing $c_1$, $\delta_{1,1}$ or $\delta_{1,2}$ increases the steepness of the smile of volatilities. Whereas increasing $c_2$, $\delta_{2,2}$ or $\delta_{2,1}$ flatten the curve. Finally, higher $\rho_1$ or $\rho_2$ respectively lowers or raises the slope of the smile.

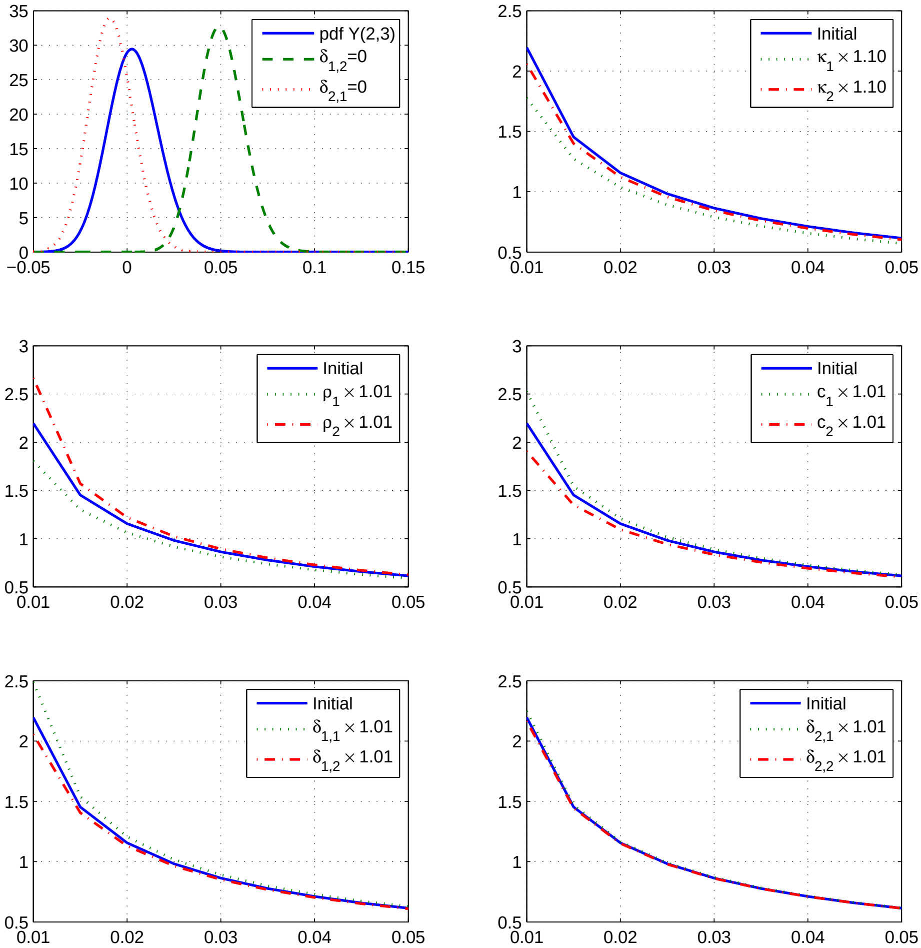


Figure 6

# 5 Conclusion

The literature provides a great deal of evidence that liquidity shortages are caused by a disequilibrium between the supply and demand of fixed income instruments, and that it impacts level of interest rates. Directly inspired from the economic monetary theory, this work presents a new interest rate model based on recent developments in the study of market micro-structure. The novelty of this approach is to consider that aggregate supply and demand of fixed income products are ruled by a bivariate Hawkes process. This introduces both path dependence and mutual excitation in the arrivals processes of bid and ask orders for interest rate products. Furthermore, quasi explicit formulas are available for moments of intensities and bond prices and changes of measure.

The econometric analysis of the one year swap rate over a period of 10 years, suggests that intensities of bid/ask orders arrivals are key factors to understand the fluctuations of rates. In particular, a negative difference between bid and ask frequencies is a solid indicator to detect liquidity shortfalls. On another hand, combining the econometric calibration with the analysis of past swap curves, allows us to filter risk premiums of processes representing the demand and supply of bonds. The distance between these risk premiums explains the steepness of the yield curve and is particularly small during the 2008 crisis. Finally, the different sensitivity analysis developed in this work, confirm that the model is tractable for derivatives pricing or for risk management purposes.

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